

MEMORIAS DE LA SOCIEDAD MATEMÁTICA MEXICANA

ISSN: 2992-8427

2025

EDITORES

Marcelo Aguilar
Luis Hernández Lamonedá

Volumen 9
octubre 2025

MEMORIAS DE LA SOCIEDAD MATEMÁTICA MEXICANA

2025

COMITÉ EDITORIAL DE LAS MEMORIAS DE LA SOCIEDAD MATEMÁTICA MEXICANA

Marcelo Aguilar

Instituto de Matemáticas, UNAM
Circuito Exterior
Ciudad Universitaria
Coyoacán, Ciudad de México, 04510
marcelo@matem.unam.mx

Luis Hernández Lamonedá

Centro de Investigación en Matemáticas (CIMAT)
Callejón de Jalisco s/n,
Mineral de Valenciana
Guanajuato 36240, Gto.
lamonedá@cimat.mx

MEMORIAS DE LA SOCIEDAD MATEMÁTICA MEXICANA

2025

EDITORES

Marcelo Aguilar
Luis Hernández Lamonedá

Volumen 9
octubre 2025

MEMORIAS DE LA SOCIEDAD MATEMÁTICA MEXICANA, Año 9, No. 9, de octubre 2025 – octubre 2026 es una publicación anual, editada por la Universidad Nacional Autónoma de México, Av. Universidad 3000, Col. Ciudad Universitaria, Alcaldía de Coyoacán, C.P. 04510, Ciudad de México, y la Sociedad Matemática Mexicana, A.C., Vicente Beristain No. 165 B, Col. Ampliación Asturias, Alcaldía de Cuauhtémoc, C.P. 06890, Ciudad de México, a través del Instituto de Matemáticas, Área de la Investigación Científica, Circuito Exterior S/N, Ciudad Universitaria, Coyoacán, C.P. 04510, Ciudad de México, Teléfonos: (+52) (55) 5622 4494 y 4496, <https://memorias-smm.matem.unam.mx/index.html>, marcelo@matem.unam.mx. Editores responsables: Marcelo Aguilar y Luis Hernández Lamonedá. Número de reserva de Derechos de Autor otorgado por INDAUTOR: 04-2017-111017154200-102, **ISSN: 2992-8427**, ambos otorgados por el Instituto Nacional del Derecho de Autor. Responsable de la última actualización de este número, Instituto de Matemáticas, Dr. Marcelo Aguilar, Área de la Investigación Científica, Circuito Exterior S/N, Ciudad Universitaria, Coyoacán, C.P. 04510, Ciudad de México, fecha de la última modificación, 26 de octubre de 2025.

El contenido de los artículos es responsabilidad de los autores y no refleja el punto de vista de los árbitros, de los Editores, de la UNAM o de la SMM. Se autoriza la reproducción total o parcial de los textos aquí publicados siempre y cuando se cite la fuente completa y la dirección electrónica de la publicación.

CONTENIDO

Prefacio

Artículos de Investigación 1

Hugo Reyna-Castañeda and María de los Ángeles Sandoval-Romero
A variational method to calculate probabilities 3

Artículos de Exposición 27

M. C. Rodríguez-Vallarte, G. Salgado
Double extensions of Lie algebras are more common than you think 29

PREFACIO

La colección *Memorias de los Congresos Nacionales de la SMM* empezó a ser publicada por la propia Sociedad Matemática Mexicana dentro de la serie *Aportaciones Matemáticas*, en 1986 (los primeros cuatro volúmenes con el título de *Programa*). Los editores de cada volumen han sometido todos los artículos recibidos a un riguroso proceso de arbitraje. Los artículos de investigación, así como los artículos de exposición, han sido revisados por matemáticos de reconocido prestigio tanto nacionales como extranjeros.

Los artículos aceptados, pero cuyo proceso de arbitraje y corrección no ha terminado cuando un volumen es enviado a la imprenta, son publicados en el volumen del año siguiente. Esto hace que las *Memorias* tengan las características de una revista internacional que publica un volumen cada año.

Por este motivo, el Comité Editorial de *Aportaciones Matemáticas* decidió registrar esta colección como publicación periódica, con el título *Memorias de la Sociedad Matemática Mexicana*, a partir de 2005.

Memorias de la Sociedad Matemática Mexicana ha sido, de hecho, una publicación periódica conjunta entre la Sociedad Matemática Mexicana y la UNAM a través del Instituto de Matemáticas. Por esta razón, en 2017 se decidió registrarla formalmente como publicación conjunta, periódica e independiente ante el Instituto Nacional del Derecho de Autor (INDAUTOR). En consecuencia, *Memorias de la Sociedad Matemática Mexicana* ya no es parte de la colección *Aportaciones Matemáticas*. El volumen correspondiente a 2005, año en que se convirtió en publicación periódica, es ahora el volumen 1.

A partir de 2021, *Memorias de la Sociedad Matemática Mexicana* se publica exclusivamente en soporte electrónico en esta página: <http://memorias-smm.matem.unam.mx/>, conservando todas sus características de cuidadosa edición, y permitiendo, a la vez, una difusión más amplia y accesible para la comunidad interesada. Se podrán consultar y descargar números anteriores (por lo pronto, están los volúmenes desde 2017 a la fecha).

Septiembre 2025

El Comité Editorial

Artículos de Investigación

A VARIATIONAL METHOD TO CALCULATE PROBABILITIES

HUGO GUADALUPE REYNA-CASTAÑEDA AND MARÍA DE LOS ÁNGELES
 SANDOVAL-ROMERO

ABSTRACT. In this paper, we prove the existence and uniqueness of the conditional expectation of an event A given a σ -algebra $\mathcal{G}$ as a linear problem in Lebesgue spaces L^p associated with a probability space through the Riesz Representation Theorems. For the L^2 case, we state Dirichlet's principle. Then, we extend this principle to specific values of p , defining the existence of the conditional expectation as a variational problem. We conclude with a proof of the Law of Total Probability using these tools.

CONTENTS

1. Introduction	4
2. Problem statement	6
3. Problem in L^2	8
4. Problem in L^p	11
5. Proof of the Law of Total Probability	14
6. Conclusions	17
A. Banach spaces	17
B. Probability spaces	19
C. Differentiable functionals	23
References	26

2020 *Mathematics Subject Classification.* 46N30, 49J50, 49J99.

Keywords and phrases. Conditional Expectation, Total Probability Law, Riesz Representation Theorems, Dirichlet Problem.

1. INTRODUCTION

The Law of Total Probability (see [4, Theorem 3.4]) is a fundamental result of classical probability theory with multiple applications in various disciplines such as physics, medicine, biology, etc. This theorem allows for the calculation of an event's probability through the computation of conditional probabilities.

Theorem 1.1 (Law of Total Probability). *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{B} = \{B_1, \dots, B_N\}$ be a collection on $\mathcal{F}$ such that $\Omega = \bigcup_{j=1}^N B_j$, $B_i \cap B_j = \emptyset$ for all $i \neq j$ and $P(B_j) > 0$ for all $j = 1, \dots, N$. Then, for any $A \in \mathcal{F}$, we have*

$$P(A) = \sum_{j=1}^N P(A|B_j)P(B_j),$$

where $P(A)$ is the total probability of event A .

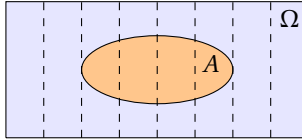


Figure 1. Law of total probability

The main concept of Theorem 1.1 is the conditional probability. We define and denote the conditional probability of event A given another event B with $P(B) > 0$ by

$$P(A|B) := \frac{P(A \cap B)}{P(B)}.$$

Conditional probability is a concept that allows us to interact with the reality of random phenomena by incorporating additional information about them.

Moreover, Theorem 1.1 is crucial to understanding how probabilities propagate under different conditions, and its proof is straightforward, since it relies on the basic properties of the probability measure. However, its essence belongs to a more general concept in probability theory: *conditional expectation* (see [4, Definition 23.4]).

Definition 1.2. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X : \Omega \rightarrow \mathbb{R}$ (Lebesgue) integrable random variable, and $\mathcal{G}$ σ -algebra such that $\mathcal{G} \subset \mathcal{F}$. The conditional expectation of X given $\mathcal{G}$ is the unique function $E(X|\mathcal{G}) : \Omega \rightarrow \mathbb{R}$ with the following properties:*

- (i) $E(X | \mathcal{G})$ is $\mathcal{G}$ -measurable.
- (ii) $E(X | \mathcal{G})$ is (Lebesgue) integrable.
- (iii) For all $B \in \mathcal{G}$,

$$\int_B X dP = \int_B E(X | \mathcal{G}) dP.$$

Conditional expectation is fundamental for understanding the average behavior of a random variable, given certain conditions or additional information. This concept is closely related to the elementary ideas of probability. For the simplest case, when the σ -algebra $\mathcal{G}$ consists only of the empty set $\emptyset$ and the entire sample space Ω , $E(X | \mathcal{G})$ becomes a constant function equal to the unconditional expectation $E(X | \mathcal{G}) = E(X)$. Consequently, if X is an indicator function 1_A for an event $A \in \mathcal{F}$, then $E(1_A | \mathcal{G})$ equals the probability of A , $P(A)$. The above suggests that the basic concept of conditional probability can be derived from conditional expectation. To do this, we need to carefully study a certain class of σ -algebras.

The classical proof of the concept of conditional expectation is based on the Radon–Nikodym theorem. The idea is to take an integrable random variable X and define a finite signed measure Q on the sub σ -algebra $\mathcal{G}$ by $Q(B) := \int_B X dP$. Someone then shows that $Q(B) = 0$ whenever $P(B) = 0$ and therefore $E(X | \mathcal{G})$ is the Radon–Nikodym derivative of Q with respect to P restricted to $\mathcal{G}$ (see [4, Theorem 28.3]).

The general aim of this work is to present a new proof of the existence and uniqueness of the conditional expectation for indicator functions of sets in a σ -algebra, using techniques from nonlinear analysis in Banach spaces. In particular, we show that the conditional expectation coincides with the unique critical point of a functional defined on suitable Banach spaces constructed from the integral on the underlying probability space. As a consequence of this result, we also obtain a new proof of the law of total probability. For this reason, we have entitled this work “A variational method to calculate probabilities.”

In Section 2, we focus on providing an overview of the problem to be solved, that is, we will frame the existence and uniqueness of the conditional expectation of an event A as a linear problem in a vector space of infinite dimension.

In Section 3, we restrict the problem to the Hilbert space L^2 of square-integrable random variables, where the inner product is the expectation of the product. This restriction allows us to invoke the Riesz–Fréchet representation theorem, and thus provide a geometric description of conditional expectation. Moreover, in the same section we cast the existence of the conditional expectation as the existence of a critical

point (indeed, a minimizer) of an energy functional defined on an infinite-dimensional Hilbert space; this is the Dirichlet principle, whose solution is furnished by the Riesz-Fréchet theorem. Although it is common in the literature (see, e.g., [4, Chapter 23]) to prove existence in L^2 via this description, we point out that the phenomenon does not depend on the space being Hilbert. Furthermore, in Section 4 we show that for certain values of $p \in (1, \infty)$ the geometry of L^p retains the key features of the L^2 case. To this end, we present a suitable version of the Riesz representation theorem and, leveraging Clarkson's inequalities together with the Hölder's inequality, extend the Dirichlet principle approach to those values of p .

Finally, in Section 5, where we prove the Law of Total Probability (Theorem 1.1) using the tools developed in the previous sections.

We gather the notation, preliminaries, and technical tools used throughout the paper in Appendices A, B, and C which we include because the main proof draws from the outset on solid results from nonlinear analysis in Banach spaces that standard probability textbooks rarely present in the requisite detail. Appendix A reviews functional-analytic notions; Appendix B summarizes the probabilistic setting, and Appendix C gathers differentiability concepts for functionals on Banach and Hilbert spaces. Where helpful, we include brief, self-contained proofs tailored to the paper's variational approach, with the aim of making the article self-contained, unifying notation, reducing external prerequisites, and clarifying the role these techniques play in extending the Dirichlet principle beyond the Hilbert framework.

Remark 1.3. *In this text, we consider two random variables identical if they are equal except on a set of probability zero. That is, $X = Y$ if $P(X = Y) = P(\{\omega \in \Omega : X(\omega) = Y(\omega)\}) = 1$.*

2. PROBLEM STATEMENT

We outline the main idea of the problem addressed in this paper. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{G} \subset \mathcal{F}$ be a σ -algebra. Restricting P to $\mathcal{G}$ we obtain a probability measure on $(\Omega, \mathcal{G})$; hence $(\Omega, \mathcal{G}, P)$ is a probability space.

Fix $A \in \mathcal{F}$. Our goal is to prove the existence and uniqueness of the conditional expectation $E(1_A | \mathcal{G})$. More precisely, we seek a unique function $\xi \in L^1(\Omega, \mathcal{G}, P)$ such that:

$$(2.1) \quad \int_{\Omega} 1_A \cdot 1_B dP = \int_{\Omega} \xi \cdot 1_B dP \quad \forall B \in \mathcal{G}.$$

Note that $1_B \in L^1(\Omega, \mathcal{G}, P)$ for all $B \in \mathcal{G}$ because $\int_{\Omega} 1_B dP = P(B) \leq 1$. Suppose $\xi \in L^1(\Omega, \mathcal{G}, P)$ is a conditional expectation of 1_A given $\mathcal{G}$. Then, by linearity of the integral, for every simple $\mathcal{G}$ measurable random variable $X = \sum_{j=1}^N \alpha_j 1_{A_j} \in L^1(\Omega, \mathcal{G}, P)$,

$$(2.2) \quad \int_{\Omega} \xi \cdot X dP = \sum_{j=1}^N \alpha_j \int_{\Omega} \xi \cdot 1_{A_j} dP = \sum_{j=1}^N \alpha_j \int_{\Omega} 1_A \cdot 1_{A_j} dP = \int_{\Omega} 1_A \cdot X dP.$$

Now, let $X \in L^1(\Omega, \mathcal{G}, P)$. There exists a sequence (X_k) of $\mathcal{G}$ -measurable simple random variables with $X_k \rightarrow X$ in $L^1(\Omega, \mathcal{G}, P)$. Passing to a subsequence (still denoted X_k), we may assume $X_k \rightarrow X$ and $|X_k| \leq Y$ almost surely for some $Y \in L^1(\Omega, \mathcal{G}, P)$ (see [4, Theorem 22.6] and [4, Definition 23.5]). By the dominated convergence theorem (see [4, Theorem 9.1]) and (2.2),

$$\int_{\Omega} \xi \cdot X dP = \lim_{k \rightarrow \infty} \int_{\Omega} \xi \cdot X_k dP = \lim_{k \rightarrow \infty} \int_{\Omega} 1_A \cdot X_k dP = \int_{\Omega} 1_A \cdot X dP.$$

Thus, if $\xi = E(1_A | \mathcal{G})$, then

$$(2.3) \quad \int_{\Omega} 1_A \cdot X dP = \int_{\Omega} \xi \cdot X dP \quad \forall X \in L^1(\Omega, \mathcal{G}, P).$$

Conversely, if $\xi \in L^1(\Omega, \mathcal{G}, P)$ satisfies (2.3), then it satisfies (2.1) in particular, and by uniqueness $\xi = E(1_A | \mathcal{G})$.

Consequently, constructing the conditional expectation reduces to finding a unique $\xi \in L^1(\Omega, \mathcal{G}, P)$ such that

$$\int_{\Omega} 1_A \cdot X dP = \int_{\Omega} \xi \cdot X dP \quad \forall X \in L^1(\Omega, \mathcal{G}, P).$$

Define $T : L^1(\Omega, \mathcal{G}, P) \rightarrow \mathbb{R}$ by

$$T(X) := \int_{\Omega} 1_A \cdot X dP.$$

This map is linear and satisfies $|T(X)| \leq \int_{\Omega} |X| dP$ for every $X \in L^1(\Omega, \mathcal{G}, P)$, so T is continuous with $\|T\|_{(L^1(\Omega, \mathcal{G}, P))^*} \leq 1$ (see Theorem B.8). Thus, we precisely formulate the problem to be solved as follows:

Problem 2.1. *Given $A \in \mathcal{F}$, does there exist a unique function $\xi \in L^1(\Omega, \mathcal{G}, P)$ such that*

$$(2.4) \quad T(X) = \int_{\Omega} \xi \cdot X dP \quad \text{for every } X \in L^1(\Omega, \mathcal{G}, P)?$$

From a conceptual point of view, the preceding analysis falls within the framework of a Riesz-type representation. In the sections that follow, we show that this approach, developed in the setting of Lebesgue spaces, solves the preceding problem and rigorously establishes the existence and uniqueness of conditional expectation. In doing so, we provide a variational characterization of the conditional expectation.

3. PROBLEM IN L^2

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $A \in \mathcal{F}$, and let $\mathcal{G}$ be a σ -algebra such that $\mathcal{G} \subset \mathcal{F}$.

Consider $(\Omega, \mathcal{G}, P)$, the probability space introduced in Section 2, and define

$$L^2(\Omega, \mathcal{G}, P) := \{X : \Omega \rightarrow \mathbb{R} : X \text{ is } \mathcal{G}\text{-measurable and } E(|X|^2) < \infty\}.$$

It is immediate that $X \in L^2(\Omega, \mathcal{G}, P)$ if and only if $|X|^2 \in L^1(\Omega, \mathcal{G}, P)$. Hence $L^2(\Omega, \mathcal{G}, P)$ is a vector space over $\mathbb{R}$.

The objective of this section is to derive, within the $L^2(\Omega, \mathcal{G}, P)$ framework, the conditional expectation of 1_A with respect to $\mathcal{G}$, in accordance with the discussion in Section 2.

The fundamental tool is Hölder's inequality [1, Theorem 4.6], which will allow us to address the problem in this setting.

For $X \in L^2(\Omega, \mathcal{G}, P)$ set

$$\|X\|_2 := (E(|X|^2))^{1/2}.$$

Theorem 3.1 (Hölder's inequality). *If $X, Y \in L^2(\Omega, \mathcal{G}, P)$, then $X \cdot Y \in L^1(\Omega, \mathcal{G}, P)$ and $\|X \cdot Y\|_1 \leq \|X\|_2 \|Y\|_2$.*

Corollary 3.2. *If $X \in L^2(\Omega, \mathcal{G}, P)$, then $X \in L^1(\Omega, \mathcal{G}, P)$ and $\|X\|_1 \leq \|X\|_2$.*

On $L^2(\Omega, \mathcal{G}, P)$ define

$$\langle X, Y \rangle_2 := E(X \cdot Y) = \int_{\Omega} X \cdot Y \, dP.$$

By Theorem 3.1 this is a well-defined inner product and satisfies

$$\langle X, X \rangle_2 = \|X\|_2^2.$$

In fact, by the Riesz–Fischer theorem [1, Theorem 4.8], $L^2(\Omega, \mathcal{G}, P)$ endowed with $\langle \cdot, \cdot \rangle_2$ is a Hilbert space.

We now recast Problem 2.1 in $L^2(\Omega, \mathcal{G}, P)$. Define $T : L^2(\Omega, \mathcal{G}, P) \rightarrow \mathbb{R}$ by

$$(3.1) \quad T(X) := \int_{\Omega} 1_A \cdot X \, dP.$$

Clearly T is linear. By Theorem 3.1, $|T(X)| \leq \|X\|_2$ for all $X \in L^2(\Omega, \mathcal{G}, P)$, hence T is continuous (see Definition A.2). The question is: Does there exist a unique function $\xi \in L^2(\Omega, \mathcal{G}, P)$ such that $T(X) = \int_{\Omega} \xi \cdot X \, dP$ for all $X \in L^2(\Omega, \mathcal{G}, P)$?

In order to give a response we have to remember that, in Hilbert spaces, continuous linear functionals are represented via the inner product by the Riesz–Fréchet representation theorem [5, Theorem 3.8-1]:

Theorem 3.3 (Riesz–Fréchet representation theorem). *Let H be a real Hilbert space and $L : H \rightarrow \mathbb{R}$ a continuous linear functional. Then there exists a unique $u \in H$ such that*

$$L(v) = \langle u, v \rangle \quad \forall v \in H.$$

By Theorem 3.3, there exists a unique $\xi \in L^2(\Omega, \mathcal{G}, P)$ such that

$$T(X) = \langle \xi, X \rangle_2 = \int_{\Omega} \xi \cdot X \, dP \quad \forall X \in L^2(\Omega, \mathcal{G}, P).$$

By Corollary 3.2, $\xi \in L^1(\Omega, \mathcal{G}, P)$, and since $1_B \in L^2(\Omega, \mathcal{G}, P)$ for every $B \in \mathcal{G}$,

$$T(1_B) = \int_{\Omega} \xi \cdot 1_B \, dP \quad \forall B \in \mathcal{G}.$$

Consequently, $\xi = E(1_A | \mathcal{G})$.

Then, we have presented a new approach to establishing the existence and uniqueness of $E(1_A | \mathcal{G})$. Moreover, the strategy will allow us to assert that $E(1_A | \mathcal{G})$ is the unique critical point (indeed, the minimizer) of an energy functional $J : L^2(\Omega, \mathcal{G}, P) \rightarrow \mathbb{R}$, an instance of the Dirichlet principle (see [1, Proposition 8.15]).

The energy functional defined in $L^2(\Omega, \mathcal{G}, P)$ is the function $J : L^2(\Omega, \mathcal{G}, P) \rightarrow \mathbb{R}$ given by

$$(3.2) \quad J(X) := \frac{1}{2} \int_{\Omega} |X|^2 \, dP - \int_{\Omega} 1_A \cdot X \, dP.$$

If J is of class $\mathcal{C}^1$ and ξ_0 is a critical point of J in $L^2(\Omega, \mathcal{G}, P)$, then

$$J'_{\xi_0}(X) = \langle \xi_0, X \rangle_2 - T(X) = 0 \quad \forall X \in L^2(\Omega, \mathcal{G}, P).$$

In particular, letting $\xi := E(1_A | \mathcal{G}) \in L^2(\Omega, \mathcal{G}, P)$, we have $\int_{\Omega} \xi \cdot 1_B = T(1_B)$ for any $B \in \mathcal{G}$ and thus $\int_{\Omega} \xi \cdot Z = T(Z)$ for any simple random variable $Z \in L^2(\Omega, \mathcal{G}, P)$, and

consequently, $\int_{\Omega} \xi \cdot X = T(X)$ for any $X \in L^2(\Omega, \mathcal{G}, P)$ because there exists a sequence of simple random variables (Z_k) such that $Z_k \rightarrow X$ in $L^2(\Omega, \mathcal{G}, P)$ (see [4, Theorem 17.2 and Theorem 17.4]). We thus obtain:

Theorem 3.4 (Dirichlet's principle). *The functional J in (3.2) is of class $\mathcal{C}^2$, and $E(1_A | \mathcal{G})$ is its unique critical point in $L^2(\Omega, \mathcal{G}, P)$; in addition, it is the minimizer of J .*

Proof. We have $J(X) = \frac{1}{2} \|X\|_2^2 - T(X)$ with T linear and continuous as in (3.1). Hence J is a $\mathcal{C}^2$ functional and, for $X \in L^2(\Omega, \mathcal{G}, P)$,

$$J'_X(Y) = \langle X, Y \rangle_2 - T(Y) \quad \forall Y \in L^2(\Omega, \mathcal{G}, P).$$

By Theorem 3.3, $\xi := E(1_A | \mathcal{G})$ is the unique element of $L^2(\Omega, \mathcal{G}, P)$ such that $\langle \xi, Y \rangle_2 - T(Y) = 0$ for all Y . Thus, $J'_\xi(Y) = 0$ for every $Y \in L^2(\Omega, \mathcal{G}, P)$. Moreover,

$$J''_\xi(Y, Y) = \langle Y, Y \rangle_2 = E(|Y|^2) > 0 \quad \text{for all } Y \neq 0,$$

so ξ is a minimizer by Theorem C.3. □

The question now is, how do we interpret $E(1_A | \mathcal{G})$ through the proof of Theorem 3.3?

In [5], the idea of the proof of Theorem 3.3 is: Since $L : H \rightarrow \mathbb{R}$ is a linear and continuous function, then $\ker L = L^{-1}(\{0\})$ is a closed vector subspace of H and, thus, $H = \ker L \oplus (\ker L)^\perp$ (see Figure 2).

If L is trivial, then $\ker L = H$ and $u = 0_H$ is a vector zero. If L is nontrivial, then $(\ker L)^\perp \neq \{0\}$. Fix $w \in \ker L$ with $\|w\|_H = 1$ and defining $u := L(w)w \in H$ yields the result.

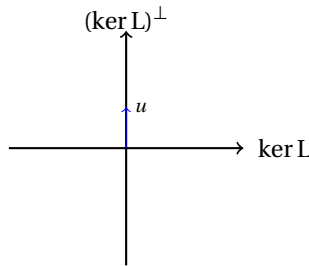


Figure 2. Orthogonal decomposition of H in the kernel of L

That is, we can interpret $E(1_A | \mathcal{G})$ as an element of $(\ker T)^\perp$.

However, in the proof of Theorem 3.3 given in [1, Theorem 5.5], two important tools are used: the Hahn-Banach theorem [1, Corollary 1.8] and the reflexivity of Hilbert spaces (see Corollary A.7).

We then ask: Is orthogonality necessary for our problem? Is there a similar theorem in spaces that are not necessarily Hilbert spaces? In the following section, we will answer these questions.

4. PROBLEM IN L^p

Again, let $(\Omega, \mathcal{F}, P)$ be a probability space, $A \in \mathcal{F}$, and let $\mathcal{G}$ be a σ -algebra with $\mathcal{G} \subset \mathcal{F}$.

Orthogonality in the Hilbert space L^2 is a classical route for introducing conditional expectation, since $E(X | \mathcal{G})$ is identified with the orthogonal projection of X onto the closed subspace $L^2(\Omega, \mathcal{G}, P)$ (see [4, Theorem 22.6] and [4, Definition 23.5]). In this section, we will show that the case $X = 1_A$ can be treated in the Lebesgue spaces $L^p(\Omega, \mathcal{G}, P)$, for $1 < p < \infty$, without appealing to the Hilbert space structure: we will derive $E(1_A | \mathcal{G})$ from the geometric properties of L^p and from the Riesz representation theorem in L^p . We will also extend the Dirichlet principle for certain values of $p \in (1, \infty)$, thus recovering that $E(1_A | \mathcal{G})$ is the unique critical point of an energy functional in L^p .

For $p \in (1, \infty)$,

$$L^p(\Omega, \mathcal{G}, P) := \{X : \Omega \rightarrow \mathbb{R} : X \text{ is } \mathcal{G}\text{-measurable and } E(|X|^p) < \infty\}.$$

It is clear that $X \in L^p(\Omega, \mathcal{G}, P)$ if and only if $|X|^p \in L^1(\Omega, \mathcal{G}, P)$. Hence $L^p(\Omega, \mathcal{G}, P)$ is a vector space over $\mathbb{R}$ [1, Theorem 4.7].

For $X \in L^p(\Omega, \mathcal{G}, P)$ we set

$$\|X\|_p := (E(|X|^p))^{1/p}.$$

Let $p \in (1, \infty)$ and denote by $q := \frac{p}{p-1} \in (1, \infty)$ the conjugate exponent, so that $\frac{1}{p} + \frac{1}{q} = 1$.

There is a general version of Hölder's inequality [1, Theorem 4.6], which we state as follows:

Theorem 4.1 (Hölder's inequality in L^p). *If $p, q \in (1, \infty)$ are conjugate exponents, $X \in L^p(\Omega, \mathcal{G}, P)$, and $Y \in L^q(\Omega, \mathcal{G}, P)$, then $X \cdot Y \in L^1(\Omega, \mathcal{G}, P)$ and $\|X \cdot Y\|_1 \leq \|X\|_p \|Y\|_q$.*

Corollary 4.2. *Let $1 \leq r < s < \infty$. Then $L^s(\Omega, \mathcal{G}, P) \subset L^r(\Omega, \mathcal{G}, P)$. Moreover, for any $X \in L^s(\Omega, \mathcal{G}, P)$ we have $\|X\|_r \leq \|X\|_s$.*

The Riesz-Fischer theorem [1, Theorem 4.8] guarantees that the space $L^p(\Omega, \mathcal{G}, P)$ with the norm $\|\cdot\|_p$ is a Banach space for all $p \in (1, \infty)$.

Note that the identity that allowed us to conclude that Hilbert spaces are reflexive through the Milman-Pettis theorem (see Theorem A.6) was the parallelogram identity and, in fact, it was sufficient to have a certain inequality. In spaces L^p , there are inequalities that similarly allow us to conclude that they are reflexive spaces [1, Theorem 4.10]. These inequalities are known as the first and second Clarkson inequalities, and for their proof, we suggest consulting [6] and [1, Theorem 4.10].

Theorem 4.3 (Clarkson inequalities). *Let $p \in (1, \infty)$ and $X, Y \in L^p(\Omega, \mathcal{G}, P)$.*

(1) *If $p \in (1, 2]$, then*

$$\left\| \frac{X+Y}{2} \right\|_p^{p/(p-1)} + \left\| \frac{X-Y}{2} \right\|_p^{p/(p-1)} \leq \left(\frac{1}{2} \|X\|_p^p + \frac{1}{2} \|Y\|_p^p \right)^{1/(p-1)}.$$

(2) *If $p \in [2, \infty)$, then*

$$\left\| \frac{X+Y}{2} \right\|_p^p + \left\| \frac{X-Y}{2} \right\|_p^p \leq \frac{1}{2} (\|X\|_p^p + \|Y\|_p^p).$$

Note that for $p = 2$, the Clarkson inequalities imply the parallelogram identity in $L^2(\Omega, \mathcal{G}, P)$.

Corollary 4.4. *$L^p(\Omega, \mathcal{G}, P)$ is uniformly convex, and thus it is reflexive for all $p \in (1, \infty)$.*

Therefore, using the Hahn-Banach theorem together with reflexivity of L^p spaces, we obtain the Riesz representation theorem [1, Theorem 4.11].

Theorem 4.5 (Riesz representation theorem on L^p). *Let $p \in (1, \infty)$ and let $\Phi : L^p(\Omega, \mathcal{G}, P) \rightarrow \mathbb{R}$ be linear and continuous. Then there exists a unique $\psi \in L^q(\Omega, \mathcal{G}, P)$ such that*

$$\Phi(X) = \int_{\Omega} \psi \cdot X \, dP \quad \forall X \in L^p(\Omega, \mathcal{G}, P).$$

If $p = 2$, then $q = 2$, and Theorem 4.5 reduces to Theorem 3.3 in the Hilbert space $L^2(\Omega, \mathcal{G}, P)$.

Hence, we recast Problem 2.1 in $L^p(\Omega, \mathcal{G}, P)$. As in Section 3, given $A \in \mathcal{F}$, consider $T : L^p(\Omega, \mathcal{G}, P) \rightarrow \mathbb{R}$ defined by

$$(4.1) \quad T(X) := \int_{\Omega} 1_A \cdot X \, dP.$$

This map is clearly linear. By Hölder's inequality (Theorem 4.1), $|T(X)| \leq \|X\|_p$ for all $X \in L^p(\Omega, \mathcal{G}, P)$; hence T is linear and continuous (see Definition A.2). According to the Riesz representation theorem (Theorem 4.5), there exists a unique $\xi \in L^q(\Omega, \mathcal{G}, P)$ such that

$$T(X) = \int_{\Omega} \xi \cdot X \, dP \quad \text{for all } X \in L^p(\Omega, \mathcal{G}, P).$$

In particular, for every $B \in \mathcal{G}$, since $1_B \in L^p(\Omega, \mathcal{G}, P)$, we have $T(1_B) = \int_{\Omega} \xi 1_B \, dP$. Moreover, because $q > 1$, Corollary 4.2 yields $\xi \in L^1(\Omega, \mathcal{G}, P)$. Thus, $\xi = E(1_A | \mathcal{G})$.

The next objective is to extend the Dirichlet principle to L^p spaces for $p \in (1, \infty)$; that is, we seek a functional $F : L^p(\Omega, \mathcal{G}, P) \rightarrow \mathbb{R}$ of class $\mathcal{C}^2$ such that

$$F'_{E(1_A | \mathcal{G})}(Y) = T(Y) \quad \forall Y \in L^p(\Omega, \mathcal{G}, P),$$

where T is given in (4.1).

Suppose $p \in (1, 2)$ and let $q \in (2, \infty)$ be its conjugate exponent. By Corollary 4.2, the functional $J_q : L^q(\Omega, \mathcal{G}, P) \rightarrow \mathbb{R}$ given by

$$(4.2) \quad J_q(X) := \frac{1}{2} \int_{\Omega} |X|^2 \, dP - T(X)$$

is well defined, since $L^q(\Omega, \mathcal{G}, P) \subset L^2(\Omega, \mathcal{G}, P) \subset L^p(\Omega, \mathcal{G}, P) \subset L^1(\Omega, \mathcal{G}, P)$ when $1 < p < 2 < q < \infty$. Here we consider the restriction of T to $L^q(\Omega, \mathcal{G}, P)$, which is linear and continuous.

Theorem 4.6. *If $p \in (1, 2)$, then J_q defined in (4.2) is of class $\mathcal{C}^2$ on $L^q(\Omega, \mathcal{G}, P)$, with*

$$J'_q(X)(Y) = \int_{\Omega} X \cdot Y \, dP - T(Y) \quad \text{and} \quad J''_q(X)(Y, Z) = \int_{\Omega} Y \cdot Z \, dP$$

for all $X, Y, Z \in L^q(\Omega, \mathcal{G}, P)$.

Proof. Let $X \in L^q(\Omega, \mathcal{G}, P)$. For $Y \in L^q(\Omega, \mathcal{G}, P)$ and $t \in \mathbb{R}$,

$$\begin{aligned} J_q(X + tY) - J_q(X) &= \frac{1}{2} \int_{\Omega} (|X + tY|^2 - |X|^2) \, dP - (T(X + tY) - T(X)) \\ &= t \int_{\Omega} X \cdot Y \, dP - t T(Y) + \frac{t^2}{2} \int_{\Omega} |Y|^2 \, dP. \end{aligned}$$

Hence,

$$\lim_{t \rightarrow 0} \frac{J_q(X + tY) - J_q(X)}{t} = \int_{\Omega} X \cdot Y \, dP - T(Y),$$

so $J'_q(X)(Y) = \int_{\Omega} X \cdot Y \, dP - T(Y)$. If $X_k \rightarrow X$ in L^q , then by Hölder's inequality

$$|J'_q(X_k)(Y) - J'_q(X)(Y)| \leq \int_{\Omega} |X_k - X| |Y| \, dP \leq \|X_k - X\|_q \|Y\|_p,$$

which shows that J'_q is continuous and thus J_q is a $\mathcal{C}^1$ functional. Moreover,

$$J'_q(X + tY)(Z) - J'_q(X)(Z) = t \int_{\Omega} Y \cdot Z \, dP,$$

whence $J''_q(X)(Y, Z) = \int_{\Omega} Y \cdot Z \, dP$. Therefore, J_q is a $\mathcal{C}^2$ functional. $\square$

Corollary 4.7. *If $p \in (1, 2)$, then $E(1_A | \mathcal{G})$ is the unique minimizer of*

$$J_q(X) := \frac{1}{2} \int_{\Omega} |X|^2 \, dP - \int_{\Omega} 1_A \cdot X \, dP$$

over $L^q(\Omega, \mathcal{G}, P)$.

Proof. According to the Riesz representation theorem (Theorem 4.5), $\xi = E(1_A | \mathcal{G})$ is the unique element of $L^q(\Omega, \mathcal{G}, P)$ such that

$$T(X) = \int_{\Omega} \xi \cdot X \, dP \quad \text{for all } X \in L^p(\Omega, \mathcal{G}, P).$$

In particular, for every $Y \in L^q(\Omega, \mathcal{G}, P)$, $T(Y) = \int_{\Omega} \xi \cdot Y \, dP$. By Theorem 4.6,

$$J'_q(\xi)(Y) = \int_{\Omega} \xi \cdot Y \, dP - T(Y) = 0 \quad \text{and} \quad J''_q(\xi)(Y, Y) = \int_{\Omega} |Y|^2 \, dP > 0$$

for all $Y \neq 0$. The conclusion follows from Theorem C.3. $\square$

5. PROOF OF THE LAW OF TOTAL PROBABILITY

Finally, let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{B} = \{B_1, \dots, B_N\}$ a finite partition of Ω with $B_j \in \mathcal{F}$ and $P(B_j) > 0$ for all $j = 1, \dots, N$. Define $\mathcal{G}$ as the σ -algebra generated by $\mathcal{B}$, that is, $\mathcal{G} := \sigma(\mathcal{B})$. In the previous sections, we have proven the existence and uniqueness of the conditional expectation of an event $A \in \mathcal{F}$ given the σ -algebra $\mathcal{G}$. We conclude this work by providing a proof of the Law of Total Probability (Theorem 1.1) using the results developed (Theorems 3.4 and 4.7). But first, we will present a simple and enlightening example of this method in the L^2 setting. And last, but not least, we

present an original proof of the Law of Total Probability that will use the version of these methods formulated for L^p spaces.

Example 5.1. *Let $N \geq 2$ and consider the finite uniform probability space*

$$\Omega = \{1, \dots, N\}, \quad \mathcal{F} = \mathcal{P}(\Omega), \quad P(\{j\}) = \frac{1}{N} \quad (j \in \Omega).$$

Fix $A \in \mathcal{F}$ and let

$$B := \{k \in \Omega : k \text{ is even}\} \quad \text{and} \quad \Omega \setminus B := \{k \in \Omega : k \text{ is odd}\},$$

so that $\{B, \Omega \setminus B\}$ is a partition of Ω , and set $\mathcal{G} := \sigma(\{B, \Omega \setminus B\})$. By Theorem B.6,

$$L^2(\Omega, \mathcal{G}, P) = \{X = \alpha 1_B + \beta 1_{\Omega \setminus B} : \alpha, \beta \in \mathbb{R}\}.$$

Define the energy functional $J : L^2(\Omega, \mathcal{G}, P) \rightarrow \mathbb{R}$ by

$$J(X) = \frac{1}{2} \int_{\Omega} |X|^2 dP - \int_{\Omega} 1_A \cdot X dP.$$

According to Theorem 3.4, $E(1_A | \mathcal{G})$ is the unique critical point of J in $L^2(\Omega, \mathcal{G}, P)$. We therefore seek $\alpha^, \beta^* \in \mathbb{R}$ such that, for*

$$\xi := \alpha^* 1_B + \beta^* 1_{\Omega \setminus B},$$

one has

$$J'_{\xi}(Y) = \int_{\Omega} \xi \cdot Y dP - \int_{\Omega} 1_A \cdot Y dP = 0 \quad \forall Y \in L^2(\Omega, \mathcal{G}, P).$$

In fact, it suffices to impose the first-order condition on the generators:

$$(5.1) \quad J'_{\xi}(1_B) = \int_{\Omega} \xi \cdot 1_B dP - \int_{\Omega} 1_A \cdot 1_B dP = 0,$$

$$(5.2) \quad J'_{\xi}(1_{\Omega \setminus B}) = \int_{\Omega} \xi \cdot 1_{\Omega \setminus B} dP - \int_{\Omega} 1_A \cdot 1_{\Omega \setminus B} dP = 0.$$

Expanding,

$$(5.3) \quad J'_{\xi}(1_B) = \alpha^* P(B) - P(A \cap B),$$

$$(5.4) \quad J'_{\xi}(1_{\Omega \setminus B}) = \beta^* P(\Omega \setminus B) - P(A \cap (\Omega \setminus B)).$$

Thus (5.3) and (5.4) vanish if and only if

$$\alpha^* = P(A | B) \quad \text{and} \quad \beta^* = P(A | \Omega \setminus B),$$

and therefore

$$\xi = E(1_A | \mathcal{G}) = P(A | B) 1_B + P(A | \Omega \setminus B) 1_{\Omega \setminus B}.$$

Consequently,

$$P(A) = \int_{\Omega} 1_A dP = \int_{\Omega} E(1_A | \mathcal{G}) dP = P(A | B) P(B) + P(A | \Omega \setminus B) P(\Omega \setminus B),$$

which is the law of total probability for this partition.

Proof of Theorem 1.1. Fix $A \in \mathcal{F}$. Let $\mathcal{B} = \{B_1, \dots, B_N\}$ be a finite partition of Ω such that $\mathcal{G} = \sigma(\mathcal{B})$. Let $p \in (1, 2]$ and let $q := \frac{p}{p-1} \in [2, \infty)$ be its conjugate exponent. By Theorem B.6,

$$L^p(\Omega, \mathcal{G}, P) = L^q(\Omega, \mathcal{G}, P) = \left\{ X = \sum_{j=1}^N \alpha_j 1_{B_j} : \alpha_1, \dots, \alpha_N \in \mathbb{R} \right\}.$$

First, we describe $E(1_A | \mathcal{G})$. Consider the energy functional $J_q : L^q(\Omega, \mathcal{G}, P) \rightarrow \mathbb{R}$ given in (4.2). According to Corollary 4.7, we seek $\alpha_1^*, \dots, \alpha_N^* \in \mathbb{R}$ such that, for

$$\xi := \sum_{j=1}^N \alpha_j^* 1_{B_j},$$

one has

$$J'_q(\xi)(Y) = \int_{\Omega} \xi \cdot Y dP - \int_{\Omega} 1_A \cdot Y dP = 0 \quad \forall Y \in L^q(\Omega, \mathcal{G}, P).$$

In fact, it suffices to verify the condition on these generators, that is,

$$J'_q(\xi)(1_{B_i}) = \int_{\Omega} \xi \cdot 1_{B_i} dP - \int_{\Omega} 1_A \cdot 1_{B_i} dP = 0 \quad \forall i \in \{1, \dots, N\}.$$

Fix $i \in \{1, \dots, N\}$. Then

$$(5.5) \quad J'_q(\xi)(1_{B_i}) = \sum_{j=1}^N \alpha_j^* \int_{\Omega} 1_{B_j} \cdot 1_{B_i} dP - \int_{\Omega} 1_A \cdot 1_{B_i} dP = \alpha_i^* P(B_i) - P(A \cap B_i),$$

where we used that $\{B_1, \dots, B_N\}$ is a partition of Ω . Therefore, (5.5) vanishes if and only if $\alpha_i^* = P(A | B_i)$.

Consequently,

$$\xi = E(1_A | \mathcal{G}) = \sum_{j=1}^N P(A | B_j) 1_{B_j}.$$

Integrating, we obtain

$$P(A) = \int_{\Omega} 1_A dP = \int_{\Omega} E(1_A | \mathcal{G}) dP = \sum_{j=1}^N P(A | B_j) P(B_j),$$

which establishes the law of total probability. $\square$

6. CONCLUSIONS

The novelty of this work lies in the formulation of conditional expectation as the solution to a variational problem. Although the existence and properties of conditional expectation are classical and extensively documented in the literature, our contribution is to reinterpret these results through the lens of Dirichlet's principle, providing a functional perspective. This approach not only recovers the standard L^2 case, but also extends the construction naturally to certain L^p spaces. To our knowledge, this variational formulation of conditional expectation in the L^p setting does not appear in the existing literature, and we believe that it offers a new perspective on the structure of conditional expectation and its role in probability theory.

A. BANACH SPACES

Let $V = (V, \|\cdot\|_V)$ and $W = (W, \|\cdot\|_W)$ be two Banach spaces over $\mathbb{R}$.

Definition A.1. *A function $f : V \rightarrow W$ is continuous on V if, for every $u \in V$ and (u_k) sequence on V such that $\lim_{k \rightarrow \infty} \|u_k - u\|_V = 0$, then $\lim_{k \rightarrow \infty} \|f(u_k) - f(u)\|_W = 0$.*

Definition A.2. *For linear and bilinear functions, the following characterization of continuity holds:*

- (a) *A linear function $L : V \rightarrow W$ is continuous if exists a constant $C > 0$ such that $\|L(v)\|_W \leq C\|v\|_V$ for all $v \in V$.*
- (b) *A bilinear function $L : V \times V \rightarrow W$ is continuous if there exists a constant $M > 0$ such that $\|L(u, v)\|_W \leq M\|u\|_V\|v\|_V$ for all $u, v \in V$.*

The space:

$$\mathcal{B}(V, W) := \{L : V \rightarrow W : L \text{ is linear and continuous}\}$$

with norm:

$$\|L\|_{\mathcal{B}(V, W)} := \sup_{\|v\|_V \leq 1} \|L(v)\|_W$$

is a Banach space over $\mathbb{R}$ [5, Theorem 2.10-2]. If $W = \mathbb{R}$ we denote this space like $V^* := \mathcal{B}(V, \mathbb{R})$ and we call it the dual space of V . In the same way, we define the space $V^{**} := \mathcal{B}(V^*, \mathbb{R})$ and call it the bidual space of V .

If $L \in \mathcal{B}(V, W)$, then $\|L(v)\|_W \leq \|L\|_{\mathcal{B}(V, W)} \|v\|_V$ for all $\|v\|_V \leq 1$. Consequently, $\|L(v)\|_V \leq \|L\|_{\mathcal{B}(V, W)} \|v\|_V$ for all $v \in V$.

Let $v \in V$. We define function $\hat{v}: V^* \rightarrow \mathbb{R}$ by

$$\hat{v}(L) := L(v).$$

It is clear that $\hat{v}$ is linear in V^* . Note that $|L(v)| \leq \|L\|_{V^*} \|v\|_V$ for all $L \in V^*$ and therefore $|\hat{v}(L)| = |L(v)| \leq \|L\|_{V^*} \|v\|_V$ for all $L \in V^*$. This implies that $\hat{v}$ is continuous at V^* .

The following result establishes a relationship between the space V and its bidual space V^{**} [5, Lemma 4.6.2].

Theorem A.3. *The function $\mathcal{J}: V \rightarrow V^{**}$ defined by*

$$\mathcal{J}(v) := \hat{v}$$

*is linear, continuous, and $\|\mathcal{J}(v)\|_{V^{**}} = \|v\|_V$ for all $v \in V$.*

The function $\mathcal{J}$ in Theorem A.3 is called the canonical mapping of V into V^{**} and we can deduce that it is injective.

Definition A.4. *A Banach space V is said to be reflexive if the canonical mapping $\mathcal{J}$ is surjective.*

The following concept will provide us with a sufficient condition for when a space is reflexive, the Milman-Pettis theorem [1, Theorem 3.31].

Definition A.5. *A normed space $V = (V, \|\cdot\|_V)$ is said to be uniformly convex if, given an arbitrary real number $\varepsilon > 0$, there is a $\delta > 0$ such that for any $v, w \in V$ with $\|v\|_V \leq 1$, $\|w\|_V \leq 1$ and $\|v - w\|_V > \varepsilon$ we have*

$$\left\| \frac{v + w}{2} \right\|_V < 1 - \delta.$$

Essentially, a normed space is uniformly convex if its closed unit ball does not contain line segments on its boundary. The most precise example of this is a Hilbert space ([1, Proposition 5.1]).

Theorem A.6 (Milman-Pettis). *Let V be a uniformly convex Banach space. Then V is reflexive.*

Corollary A.7. *If $H = (H, \langle \cdot, \cdot \rangle, \|\cdot\|_H)$ is the Hilbert space over $\mathbb{R}$, then H is uniformly convex, and thus it is reflexive.*

B. PROBABILITY SPACES

A probability space $(\Omega, \mathcal{F}, P)$ consists of three elements:

- (a) Ω is a nonempty set.
- (b) $\mathcal{F}$ is a σ -algebra in Ω , that is, $\mathcal{F}$ is a collection of subsets of Ω such that:
 - (b.1) $\Omega \in \mathcal{F}$,
 - (b.2) If $A \in \mathcal{F}$ then $\Omega \setminus A \in \mathcal{F}$,
 - (b.3) $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$ for any countable collection $\{A_k \in \mathcal{F} : k \in \mathbb{N}\}$.
- (c) P is a probability measure, that is, $P : \mathcal{F} \rightarrow [0, 1]$ satisfies:
 - (c.1) $P(\Omega) = 1$,
 - (c.2) $P(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} P(A_k)$ whenever $\{A_k : k \in \mathbb{N}\}$ is a disjoint countable collection of members of $\mathcal{F}$.

Definition B.1. Let $\mathcal{C}$ be an arbitrary collection of subsets of Ω . We define the σ -algebra generated by $\mathcal{C}$, denoted by $\sigma(\mathcal{C})$, as the intersection of all the σ -algebras of subsets of Ω that contain $\mathcal{C}$, that is,

$$\sigma(\mathcal{C}) := \bigcap \{ \mathcal{F} : \mathcal{F} \text{ } \sigma\text{-algebra and } \mathcal{C} \subset \mathcal{F} \}.$$

$\sigma(\mathcal{C})$ is a σ -algebra of subsets of Ω and, in fact, it is the smallest σ -algebra that contains $\mathcal{C}$. Indeed, by definition, it is clear that $\mathcal{C} \subset \sigma(\mathcal{C})$ and if $\mathcal{F}_0$ is a σ -algebra such that $\mathcal{C} \subset \mathcal{F}_0$ then $\mathcal{F}_0 \in \{ \mathcal{F} : \mathcal{F} \text{ } \sigma\text{-algebra and } \mathcal{C} \subset \mathcal{F} \}$ and, thus, $\sigma(\mathcal{C}) \subset \mathcal{F}_0$.

It is possible to provide a description of the σ -algebra generated by a finite partition of Ω . This result is a very common exercise in classical textbooks of the theory, so we provide a detailed proof, which we will also use in the present work.

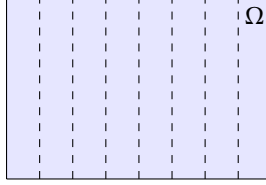
Proposition B.2. Let $\mathcal{B} = \{B_1, \dots, B_N\}$ be a finite collection of elements from $\mathcal{F}$ such that they form a partition of Ω . Then

$$\sigma(\mathcal{B}) = \left\{ \bigcup_{k \in \tau} B_k : \tau \subset \{1, \dots, N\} \text{ and } \tau \neq \emptyset \right\} \cup \{\emptyset\}.$$

Proof. We define

$$\mathfrak{A} := \left\{ \bigcup_{t \in \tau} B_t : \tau \subset \{1, \dots, N\} \text{ and } \tau \neq \emptyset \right\} \cup \{\emptyset\}.$$

It is clear that $\mathcal{B} \subset \mathfrak{A}$ and by definition of σ -algebra, $\mathfrak{A} \subset \sigma(\mathcal{B})$. We will prove that $\mathfrak{A}$ is a σ -algebra. Note that since the elements of $\mathcal{B}$ form a finite partition of Ω , $\mathfrak{A}$ is a finite set such that it suffices to prove that $\mathfrak{A}$ is closed under finite unions.

Figure 3. Partition of Ω

Given by $\Omega = B_1 \cup \dots \cup B_N$ then $\Omega \in \mathfrak{A}$. Now, let $A \in \mathfrak{A}$. It is clear that if $A = \emptyset$ then $\Omega \setminus A = \Omega \in \mathfrak{A}$ and in the same way if $A = \Omega$ then $\Omega \setminus A = \emptyset \in \mathfrak{A}$. We suppose that $A \neq \emptyset$ and that there exists a proper non-empty subset τ of $\{1, \dots, N\}$ such that

$$A = \bigcup_{t \in \tau} B_t.$$

Again, since $\Omega = \bigcup_{j \in \{1, \dots, N\}} B_j$ then

$$\Omega \setminus A = \left(\bigcup_{j \in \{1, \dots, N\}} B_j \right) \setminus \left(\bigcup_{t \in \tau} B_t \right) = \bigcup_{t \in \{1, \dots, N\} \setminus \tau} B_t \in \mathfrak{A}.$$

Finally, if $A_1, A_2 \in \mathfrak{A} \setminus \{\emptyset\}$, there exist nonempty subsets τ_1 and τ_2 of $\{1, \dots, N\}$ such that

$$A_1 = \bigcup_{i \in \tau_1} B_i \quad \text{and} \quad A_2 = \bigcup_{j \in \tau_2} B_j.$$

Therefore,

$$A_1 \cup A_2 = \left(\bigcup_{i \in \tau_1} B_i \right) \cup \left(\bigcup_{j \in \tau_2} B_j \right) = \bigcup_{t \in \tau_1 \cup \tau_2} B_t,$$

and hence $A_1 \cup A_2 \in \mathfrak{A}$. The case where $A_1 = \emptyset$ or $A_2 = \emptyset$ is clear. Consequently, $\mathfrak{A}$ is a σ -algebra in Ω such that $\mathcal{B} \subset \mathfrak{A}$ and thus $\sigma(\mathcal{B}) \subset \mathfrak{A}$. $\square$

Definition B.3. A function $X : \Omega \rightarrow \mathbb{R}$ is a random variable (or $\mathcal{F}$ -measurable function) if $X^{-1}(-\infty, c] \in \mathcal{F}$ for all $c \in \mathbb{R}$

In the previous definition, it is important to note that the measurability of the function depends on the σ -algebra considered in the probability space.

Since the main objective of this work is to present the concept of conditional expectation for certain random variables, in this section we will study some relevant examples.

Proposition B.4. If $A \in \mathcal{F}$ then indicator function 1_A given by

$$1_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A, \end{cases}$$

is a random variable.

Proof. For $c \in \mathbb{R}$ we have that

$$1_A^{-1}(-\infty, c] = \{\omega \in \Omega : 1_A(\omega) \leq c\} = \begin{cases} \Omega & \text{if } c \geq 1, \\ \Omega \setminus A & \text{if } 0 \leq c < 1, \\ \emptyset & \text{if } c < 0. \end{cases}$$

Thus, $1_A^{-1}(-\infty, c] \in \mathcal{F}$ for all $c \in \mathbb{R}$. □

Proposition B.5. A linear combination of indicator functions of sets in $\mathcal{F}$ is a random variable.

Proof. Let $X = \sum_{j=1}^N \alpha_j 1_{A_j}$ with $\alpha_j \in \mathbb{R}$ and $A_j \in \mathcal{F}$ for all $j = 1, \dots, N$ (see Figure 4).

If $c \in \mathbb{R}$ is any real number, then

$$\{\omega \in \Omega : X(\omega) \leq c\} = \begin{cases} \emptyset & \text{if } \alpha_j > c \ \forall j = 1, \dots, N, \\ \bigcup \{A_j : \alpha_j \leq c\} & \text{otherwise.} \end{cases}$$

Consequently, $X^{-1}(-\infty, c] \in \mathcal{F}$ for all $c \in \mathbb{R}$. □

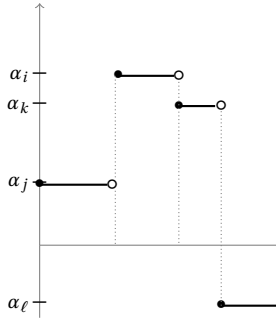


Figure 4. Linear combination of indicator functions

A function that is a linear combination of indicator functions is referred to as a simple random variable [4, Definition 9.1].

With the previous results, we can characterize the measurable functions in the σ -algebra generated by a finite partition of Ω .

Proposition B.6. *Let $\mathcal{B} = \{B_1, \dots, B_N\}$ be a finite collection of elements from $\mathcal{F}$ such that they form a partition of Ω . If $\mathcal{G} := \sigma(\mathcal{B})$ then $X : \Omega \rightarrow \mathbb{R}$ is $\mathcal{G}$ measurable if and only if for each $j = 1, \dots, N$ there exists $\alpha_j \in \mathbb{R}$ such that $X = \sum_{j=1}^N \alpha_j 1_{B_j}$.*

Proof. $\Leftarrow$: This follows from Proposition B.5.

$\Rightarrow$: Suppose, by contradiction, that there exist $j_0 \in \{1, \dots, N\}$ and $\omega_\alpha, \omega_\beta \in B_{j_0}$ such that $X(\omega_\alpha) = \alpha < \beta = X(\omega_\beta)$.

We know that X is $\mathcal{G}$ -measurable, so $X^{-1}(-\infty, \alpha] \in \mathcal{G}$ and therefore there exists $\tau \subset \{1, \dots, N\}$ such that $X^{-1}(-\infty, \alpha] = \bigcup_{t \in \tau} B_t$ by Proposition B.2. It is clear that $B_{j_0} \cap X^{-1}(-\infty, \alpha] \neq \emptyset$ and using the hypothesis, we obtain $B_{j_0} \cap X^{-1}(-\infty, \alpha] = \bigcup_{t \in \tau} (B_t \cap B_{j_0})$. However, the sets B_j form a partition of Ω so $B_{j_0} \cap X^{-1}(-\infty, \alpha] \neq \emptyset$ if and only if $B_{j_0} \cap X^{-1}(-\infty, \alpha] = B_{j_0}$. This is a contradiction, because $\omega_\beta \in B_{j_0} \setminus X^{-1}(-\infty, \alpha]$. $\square$

One of the fundamental properties of conditional expectation is formulated in terms of the Lebesgue integral, namely the equality between the integrals of a random variable and its conditional expectation over every set measurable with respect to the σ -algebra under consideration. Since the development of this work relies on this formulation, we now present some basic properties of random variables that will be useful.

Definition B.7. *If $X : \Omega \rightarrow \mathbb{R}$ is a random variable in $(\Omega, \mathcal{F}, P)$ we define the expectation using the Lebesgue integral, that is,*

$$E(X) := \int_{\Omega} X dP.$$

We say that X is (Lebesgue) integrable if $E(|X|) < \infty$.

In this paper, we consider two random variables identical if they are equal except on a set of probability zero. That is, $X = Y$ if $P(X = Y) = P(\{\omega \in \Omega : X(\omega) = Y(\omega)\}) = 1$.

We write $L^1(\Omega, \mathcal{F}, P)$ to denote the set of all integrable random variables on $(\Omega, \mathcal{F}, P)$. We state some properties of expectation whose proofs can be found in [4, Theorem 9.1].

Theorem B.8. *Let $(\Omega, \mathcal{F}, P)$ be a probability space.*

- (a) $L^1(\Omega, \mathcal{F}, P)$ is a vector space over $\mathbb{R}$, and expectation is a linear map on $L^1(\Omega, \mathcal{F}, P)$, that is, if $X, Y \in L^1(\Omega, \mathcal{F}, P)$ and $\alpha, \beta \in \mathbb{R}$ then $E(\alpha X + \beta Y) = \alpha E(X) + \beta E(Y)$.
- (b) $X \in L^1(\Omega, \mathcal{F}, P)$ if and only if $|X| \in L^1(\Omega, \mathcal{F}, P)$ and in this case $E(X) \leq E(|X|)$.

- (c) If $X, Y \in L^1(\Omega, \mathcal{F}, P)$ then $X = Y$ if and only if $E(X) = E(Y)$.
 (d) If $X, Y \in L^1(\Omega, \mathcal{F}, P)$ is such that $X \leq Y$ then $E(X) \leq E(Y)$

For $X \in L^1(\Omega, \mathcal{F}, P)$ we denote:

$$\|X\|_1 := E(|X|).$$

Thus, $L^1(\Omega, \mathcal{F}, P)$ with the norm $\|\cdot\|_1$ is a Banach space. This result is known as the Riesz-Fischer theorem, and its proof is provided in [1, Theorem 4.8].

The main result of this section establishes that every random variable can be obtained as the pointwise limit of a sequence of integrable random variables. More precisely, if X is a random variable, then there exists a sequence (X_k) of simple random variables such that $|X_k| \leq |X|$ for any $k \in \mathbb{N}$ and $X_k \rightarrow X$ on Ω (see [4, Result 1, Chapter 9]). Consequently, we obtain the following theorem (see [4, Theorem 9.1]).

Theorem B.9. *If $X \in L^1(\Omega, \mathcal{F}, P)$ then there exists a sequence (X_k) of simple random variables in $L^1(\Omega, \mathcal{F}, P)$ such that $\lim_{k \rightarrow \infty} \|X_k - X\|_1 = 0$.*

C. DIFFERENTIABLE FUNCTIONALS

In this brief appendix, we will gather concepts on the differentiability of the energy functional, which are generally developed for an arbitrary Hilbert space over $\mathbb{R}$. See [2] and [3] for more details.

Let $V = (V, \|\cdot\|)$ be a Banach space over $\mathbb{R}$, $V^* := \{L : V \rightarrow \mathbb{R} : L \text{ is linear and continuous}\}$ a dual space, and $\text{Bil}(V \times V) := \{L : V \times V \rightarrow \mathbb{R} : L \text{ is bilinear and continuous}\}$.

Definition C.1. *Let $F : V \rightarrow \mathbb{R}$ be a function.*

- (a) *If, for $u \in V$ and $v \in V$, the limit*

$$\lim_{t \rightarrow 0} \frac{F(u + tv) - F(u)}{t}$$

exists, then its value is called the derivative of F at the point u and in the direction v .

- (b) *$F : V \rightarrow \mathbb{R}$ is Gateaux-differentiable at the point $u_0 \in V$ if, for every $v \in V$, the derivative of F at the point u and in the direction v exists and the function $F'_u : V \rightarrow \mathbb{R}$ given by*

$$F'_u(v) := \lim_{t \rightarrow 0} \frac{F(u + tv) - F(u)}{t}$$

is an element of V^ . At times, to simplify the notation, we will also write $F'(u)(v)$.*

- (c) $F : V \rightarrow \mathbb{R}$ is Gateaux-differentiable on V if it is Gateaux-differentiable at every point $u \in V$. The function given by

$$F' : V \rightarrow V^*, \quad u \mapsto F'_u,$$

is called the Gateaux derivative of F .

- (d) $F : V \rightarrow \mathbb{R}$ is of class $\mathcal{C}^1$ in V if and only if F is Gateaux-differentiable on H and $F' : V \rightarrow V^*$ is continuous (see [2, Proposition 3.2.15]).
- (e) u_0 is a critical point of F in V if $F'_{u_0}(v) = 0$ for all $v \in V$.

Definition C.2. Let $F : V \rightarrow \mathbb{R}$ be a function of class $\mathcal{C}^1$.

- (a) F has a second Gateaux derivative at $u_0 \in V$ if there exists a continuous bilinear form $L : V \times V \rightarrow \mathbb{R}$ such that for every $v, w \in V$

$$\lim_{t \rightarrow 0} \frac{F'_{u_0+tv}(w) - F'_{u_0}(v) - L(tv, w)}{t} = 0.$$

We denote a second Gateaux derivative of F in u_0 by $L := F''_{u_0}$. Hence, the second Gateaux derivative on $u_0 \in V$ is given by continuous bilinear form (see [2, Remark 3.2.29])

$$F''_{u_0}(v, w) := \lim_{t \rightarrow 0} \frac{F'_{u_0+tv}(w) - F'_{u_0}(w)}{t}.$$

- (b) $F : V \rightarrow \mathbb{R}$ is twice Gateaux-differentiable on V if it has a second Gateaux derivative at every point $u \in V$. The function given by

$$F'' : V \rightarrow \text{Bil}(V \times V), \quad u \mapsto F''_u,$$

is called the second Gateaux derivative of F .

- (c) F is of class $\mathcal{C}^2$ in V if and only if F has a continuous second Gateaux derivative on V ([2, Proposition 3.2.18]).

According to the established objective, the following result provides a sufficient condition for finding minimum points of certain functionals [3, Theorem 2].

Theorem C.3. Let $F : V \rightarrow \mathbb{R}$ be a function of class $\mathcal{C}^2$. F has a minimum point $u \in V$ if $F'_u(v) = 0$ for all $v \in V$ and $F''_u(v, v) > 0$ for all $v \in V \setminus \{0_V\}$.

In the following propositions, we introduce several examples of $\mathcal{C}^2$ functions on Banach spaces, which we shall use repeatedly throughout this work.

Proposition C.4. *Fixed $x_0 \in \mathbb{R}$ and let $F : V \rightarrow \mathbb{R}$ given by $F(v) = x_0$. Then, F is of class $\mathcal{C}^2$ in V , $F'_u(v) = 0$ and $F''_u(v, w) = 0$ for all $u, v, w \in V$.*

Proof. We have that $F(u + tv) - F(u) = x_0 - x_0 = 0$ for any $u, v \in V$ and $t \in \mathbb{R}$. Thus, $F'_u(v) = 0$ for any $u, v \in V$ and consequently F is of class $\mathcal{C}^1$ on V .

On the other hand, $F'_{u+tv}(w) - F'_u(w) = 0$ for any $u, v, w \in V$ and $t \in \mathbb{R}$ so that $F''_u(v, w) = 0$ for any $u, v, w \in V$. Thus, F is of class $\mathcal{C}^2$ on V . $\square$

Proposition C.5. *If $L \in V^*$ then L is of class $\mathcal{C}^2$ in V , $L' = L$ and $L'' = 0$.*

Proof. For any $u, v \in V$ and $t \in \mathbb{R}$,

$$\frac{L(u + tv) - L(u)}{t} = L(v),$$

so $L'_u(v) = L(v)$ for all u, v . Hence L' is constant, continuous and its derivative is the zero bilinear form:

$$L''_u(v, w) = 0 \quad \forall u, v, w \in V.$$

Therefore L is of class $\mathcal{C}^2$. $\square$

Proposition C.6. *Let $H = (H, \langle \cdot, \cdot \rangle, \|\cdot\|_H)$ be a Hilbert space over $\mathbb{R}$. The function $F : H \rightarrow \mathbb{R}$ given by $F(u) = \frac{1}{2}\|u\|_H^2$ is of class $\mathcal{C}^2$ in H , $F'_u(v) = \langle u, v \rangle$ and $F''_u(v, w) = \langle v, w \rangle$ for all $u, v, w \in H$.*

Proof. Let $u \in H$. For every $v \in H$ and $t \in \mathbb{R}$ we have that:

$$F(u + tv) - F(u) = \frac{1}{2}(\|u\|_H^2 - 2t\langle u, v \rangle + t^2\|v\|_H^2 - \|u\|_H^2) = t\langle u, v \rangle + \frac{t^2}{2}\|v\|_H^2.$$

Hence,

$$\lim_{t \rightarrow 0} \frac{F(u + tv) - F(u)}{t} = \lim_{t \rightarrow 0} \left(\langle u, v \rangle - \frac{t}{2}\|v\|_H^2 \right) = \langle u, v \rangle \quad \forall v \in H.$$

That is, $F'_u(v) = \langle u, v \rangle$ for all $v \in H$ and it is clear that $F'_u \in H^*$ by Fréchet-Riesz representation theorem (see [1, Theorem 5.5]). Let (u_k) be a sequence on H such that $\|u_k - u\|_H \rightarrow 0$ as $k \rightarrow \infty$. By Cauchy-Schwarz inequality,

$$|F'_{u_k}(v) - F'_u(v)| = |\langle u_k - u, v \rangle| \leq \|u_k - u\|_H \|v\|_H \quad \forall v \in H.$$

Thus, F' is continuous and F is of class $\mathcal{C}^1$ on H .

Now, for any $v, w \in H$ and $t \in \mathbb{R}$, $F'_{u+tv}(w) - F'_u(w) = \langle u + tv, w \rangle - \langle u, w \rangle = t\langle v, w \rangle$ and consequently:

$$\lim_{t \rightarrow 0} \frac{F'_{u+tv}(w) - F'_u(w)}{t} = \langle v, w \rangle \quad \forall v, w \in H.$$

Therefore, $F''_u(v, w) = \langle v, w \rangle$ for all $v, w \in H$ and is a bilinear continuous form. This shows that F is of class $\mathcal{C}^2$. $\square$

REFERENCES

1. BREZIS, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.
2. DRÁBEK, P. AND MILOTA, J. *Methods of Nonlinear Analysis. Applications to Differential Equations*, Birkhäuser Verlag AG, Basel-Boston-Berlin, 2007.
3. GELFAND, I. M. AND FOMIN, S. V. *Calculus of Variations*, Prentice-Hall, Inc. Englewood Cliffs, New Jersey, 1963.
4. JACOD, J. AND PROTTER, P. *Probability Essentials*, Springer, Berlin, 2004.
5. KREYSZIG, E. *Introductory Functional Analysis with Applications*, John Wiley & Sons, USA, 1978.
6. RAMASWAMY, S A. A simple proof of Clarkson's inequality. *Proc. Am. Math. Soc.* **68** (1978), 249-250.
7. WILLEM, M. *Minimax Theorems*, Prog. Nonlinear Differ. Equ. Their Appl., **24**, Birkhäuser, 1996.
8. ZEIDLER, E. *Applied Functional Analysis Main Principles and Their Applications*, Springer, New York, 1995.

DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, MEXICO CITY, MEXICO

E-mail address: hugoreyna46@ciencias.unam.mx

DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, MEXICO CITY, MEXICO

E-mail address: selegna@ciencias.unam.mx

Artículos de Exposición

DOUBLE EXTENSIONS OF LIE ALGEBRAS ARE MORE COMMON THAN YOU THINK

M. C. RODRÍGUEZ-VALLARTE, G. SALGADO

ABSTRACT. The purpose of this expository survey is to demonstrate, in a self-contained manner, that the *double extension process* enables the construction of nearly all n -dimensional families of real or complex Lie algebras (where $n = 3, 4, 5$). Most of low-dimensional indecomposable solvable Lie algebras can be obtained as a double extension of a Lie subalgebra of codimension 2. In contrast, all nilpotent Lie algebras can be obtained through this process.

INTRODUCTION

The main motivation of this survey is to show how linear algebra, combined with basic knowledge of Lie algebra theory, leads to significant and non-trivial results to construct and understand families of low-dimensional Lie algebras. Specifically, we construct families of low-dimensional Lie algebras using the double extension process. This process involves extending an n -dimensional Lie algebra to a $(n + 2)$ -dimensional one by introducing new elements satisfying specific algebraic relations. Consequently, under certain conditions, the double extension process allows to provide inductive constructions of families of finite-dimensional Lie algebras. For example, all nilpotent Lie algebras can be obtained in this way.

2020 Mathematics Subject Classification. Primary: 17B, Secondary: 17-02.

Keywords and phrases. Lie algebras; Central extension; Double extension.

The theory of Lie groups and Lie algebras constitutes an important framework not only within mathematics but also across physics and, more recently, in some engineering domains. Classifying Lie algebras presents a challenging task, even in low dimensions. Various approaches and classifications exist in the literature, each guided by distinct ideas. For a comprehensive exploration of the development of these classifications see [5] and the references given there. In recent years, the double extension process has proven to be useful to obtain an inductive construction of finite-dimensional Lie algebras admitting a geometric structure. This concept was introduced by V. Kac (see [9], §2, exercise 2.10) to build finite-dimensional quadratic Lie algebras, i.e., finite-dimensional Lie algebras endowed with an invariant metric. A few years later, A. Medina y P. Revoy [10] generalized this idea to prove that every finite-dimensional quadratic indecomposable Lie algebra can be obtained as a double extension of a quadratic subalgebra by an appropriate subspace. Therefore, this fact reduces the study of finite-dimensional quadratic Lie algebras to the study of finite-dimensional indecomposable quadratic Lie algebras. Recent research has focused on double extensions for specific types of Lie algebras, such as symplectic and contact Lie algebras. For those keen on exploring further, a more thorough study of the attained results and their significance can be found in [11], [3], [13], [1], and [2].

To illustrate the usefulness of the double extension, we focus on dimensions $n = 1$ through $n = 5$ to show how most n -dimensional Lie algebras can be obtained as a double extension of a suitable $(n - 2)$ -dimensional Lie algebra. We employ the classification of low-dimensional indecomposable Lie algebras given by L. Šnobl and P. Winternitz [15], since as far as we know, it is the most recent and comprehensive. Thus, for the sake of completeness of this self-contained expository survey, the work is organized as follows: in Section §1 we provide basic concepts of Lie algebras such as nilpotency, solvability, semisimplicity and the Levi-Maltcev Theorem. Additionally, we discuss *central extensions* and *extensions by a derivation*, both of which play a crucial role in defining double extensions. We present some examples of families of Lie algebras that cannot be obtained in this way, and we also show that every element within the family of nilpotent Lie algebras can be obtained through a double extension from a Lie subalgebra of codimension 2. In Section §2 we explicitly describe low-dimensional

Lie algebras as double extensions of Lie subalgebras of codimension 2, if possible. Furthermore, we show that most of the indecomposable solvable 5-dimensional Lie algebras can be obtained through the double extension of a 3-dimensional Lie algebra.

1. BASIC CONCEPTS

Throughout this work, $\mathbb{F}$ stands either for $\mathbb{R}$ or $\mathbb{C}$; i.e. the real or complex fields. Let V be a finite-dimensional vector space over $\mathbb{F}$ and let $V^* := \text{Hom}_{\mathbb{F}}(V, \mathbb{F})$ be the associated dual vector space to V . If $\{e_1, \dots, e_n\}$ is a basis of V then $\{e^1, \dots, e^n\}$ denotes its associated dual basis in V^* , i.e., $e^i(e_j) = \delta_i^j$, for all $1 \leq i, j \leq n$, where δ_i^j is the Kronecker delta function.

A *Lie algebra* over $\mathbb{F}$ is a pair $(V, [\cdot, \cdot])$ where V is a finite-dimensional vector space over $\mathbb{F}$ and $[\cdot, \cdot]$ is a multiplication on V , i.e., $[\cdot, \cdot] : V \times V \rightarrow V$ is a bilinear function satisfying:

- (1) $[u, v] = -[v, u]$,
- (2) $[[u, v], w] + [[v, w], u] + [[w, u], v] = 0$

for all $u, v, w \in V$.

The multiplication $[\cdot, \cdot]$ is called the *Lie bracket*. Equation (1) implies that the Lie bracket is *skew-symmetric*. Equation (2) is called the *Jacobi identity* and it plays a crucial role in the theory.

From now on we follow the *usual* notation for a Lie algebra, i.e., we denote by $\mathfrak{g}$ the pair $(V, [\cdot, \cdot])$.

Given a Lie algebra $\mathfrak{g}$, a *Lie subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ such that $[x, y] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$, in other words, a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a vector subspace closed under the Lie bracket of $\mathfrak{g}$. An *ideal* $\mathfrak{h}$ of $\mathfrak{g}$ is a vector subspace such that $[x, y] \in \mathfrak{h}$ for all $x \in \mathfrak{h}$ and all $y \in \mathfrak{g}$. Clearly, if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ then $\mathfrak{h}$ is also a Lie subalgebra of $\mathfrak{g}$. However, we will see later that it is not true that every subalgebra is an ideal. Observe that $\mathfrak{h} = \{0\}$ and $\mathfrak{h} = \mathfrak{g}$ are always ideals of $\mathfrak{g}$, these ideals are called *trivial* ideals.

Examples:

- (1) Consider a vector space V over $\mathbb{F}$ and define $[u, v] = 0$ for all $u, v \in V$. In this way we obtain the *abelian* Lie algebra. If V has finite dimension, say $\dim V = n$, we denote this Lie algebra by $\mathfrak{a}_n$. Certainly every vector subspace of V is an ideal of $\mathfrak{a}_n$.
- (2) Let $V = \mathbb{R}^3$ and define $[u, v] = u \times v$ for all $u, v \in V$, that is, the Lie bracket is the cross product. A straightforward calculation shows that $(V, [\cdot, \cdot])$ is a Lie algebra that it is usually denoted by $\mathfrak{so}(3)$. If $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$, then $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$ and $[e_3, e_1] = e_2$. Using these relations one can prove that if $\mathfrak{h}$ is a non-zero ideal of $\mathfrak{so}(3)$, then $\mathfrak{h} = \mathfrak{so}(3)$, i.e., $\mathfrak{so}(3)$ has only trivial ideals. On the other hand, clearly $\mathfrak{h} = \langle e_1 \rangle$ is a 1-dimensional Lie subalgebra of $\mathfrak{so}(3)$ since $[e_1, e_1] = 0 \in \mathfrak{h}$, however it is not an ideal since $[e_1, e_2] = e_3 \notin \mathfrak{h}$.
- (3) Consider $V = \text{Mat}(n \times n, \mathbb{F})$ and define $[A, B] = AB - BA$ for all $A, B \in \text{Mat}(n \times n, \mathbb{F})$, that is, the Lie bracket of two matrices is given by its *commutator*. This Lie algebra is denoted by $\mathfrak{gl}(n)$ and it is called the *general linear* Lie algebra. If I_n denotes the identity matrix, then $[I_n, A] = 0$ for all $A \in \mathfrak{gl}(n)$ and thus $\mathfrak{h} = \langle I_n \rangle$ is an ideal of $\mathfrak{gl}(n)$. Moreover, if we define $\mathfrak{sl}(n) := \{A \in \mathfrak{gl}(n) \mid \text{Tr}(A) = 0\}$, then $\mathfrak{sl}(n)$ also is an ideal of $\mathfrak{gl}(n)$ and $\langle I_n \rangle \cap \mathfrak{sl}(n) = \{0\}$, therefore, at least as vector spaces, we have that

$$\mathfrak{gl}(n) = \langle I_n \rangle \oplus \mathfrak{sl}(n).$$

- (4) Let W be a vector space over $\mathbb{F}$. Consider $V := \text{End}(W)$ the vector space of $\mathbb{F}$ -linear endomorphisms of W , and define the Lie bracket on V as $[T, S] = T \circ S - S \circ T$ for all $T, S \in V$. Again, we obtain a Lie algebra that it is denoted by $\mathfrak{gl}(V)$. When W has finite dimension, say $\dim(W) = n$, we can fix a basis β of W and consider the *linear isomorphism* $\Phi_\beta : \text{End}(W) \rightarrow \text{Mat}(n \times n, \mathbb{F})$ assigning to T its matrix representation with respect to the basis β , i.e., $T \mapsto \Phi_\beta(T) = [T]_\beta$. Clearly the isomorphism Φ_β depends on the choice of the basis β of W . Since $[T \circ S]_\beta = [T]_\beta [S]_\beta$, it is immediate that the following equation holds:

$$\Phi_\beta([T, S]_{\mathfrak{gl}(V)}) = [\Phi_\beta(T), \Phi_\beta(S)]_{\mathfrak{gl}(n)}.$$

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras over the same field $\mathbb{F}$. A *morphism* of Lie algebras from $\mathfrak{g}$ to $\mathfrak{h}$ is a linear transformation $\varphi : \mathfrak{g} \rightarrow \mathfrak{h}$ satisfying:

$$\varphi([x, y]_{\mathfrak{g}}) = [\varphi(x), \varphi(y)]_{\mathfrak{h}}$$

for all $x, y \in \mathfrak{g}$. An invertible morphism of Lie algebras is called an *isomorphism* of Lie algebras, and in this case we say that the Lie algebras are isomorphic. Since the composition of morphisms of Lie algebras is again a morphism of Lie algebras, we can define the *group of automorphisms* of a Lie algebra as follows:

$$\text{Aut}(\mathfrak{g}) := \{\varphi : \mathfrak{g} \rightarrow \mathfrak{g} \mid \varphi \text{ is an isomorphism of Lie algebras}\}.$$

If $\varphi : \mathfrak{g} \rightarrow \mathfrak{h}$ is a morphism of Lie algebras, it can be easily verified that $\mathfrak{k} := \text{Ker}(\varphi)$ is an ideal of $\mathfrak{g}$, whereas $\mathfrak{i} := \text{Im}(\varphi)$ is a Lie subalgebra of $\mathfrak{h}$.

If V is a finite-dimensional vector space with a fixed basis β , the reader can verify that $\Phi_{\beta} : \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(n)$ defined by $\Phi_{\beta}(T) = [T]_{\beta}$ is an isomorphism of Lie algebras.

Given a Lie algebra $\mathfrak{g}$ over the field $\mathbb{F}$, a linear transformation $D \in \text{End}(\mathfrak{g})$ is a *derivation* of $\mathfrak{g}$ if $D([x, y]) = [D(x), y] + [x, D(y)]$ for all $x, y \in \mathfrak{g}$. It is easy to show that the composition of two derivations is not a derivation, however their *commutator* is a derivation. Therefore,

$$\text{Der}(\mathfrak{g}) := \{D \in \text{End}(\mathfrak{g}) \mid D([x, y]) = [D(x), y] + [x, D(y)]\}$$

is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$ (c.f. Example (4) above). Moreover, for $x \in \mathfrak{g}$ let $\text{ad}(x)$ denote the linear transformation in $\mathfrak{g}$ defined by $\text{ad}(x)(y) := [x, y]$ for all $y \in \mathfrak{g}$, then it is a simple matter to check that the following statements are equivalent: (a) $\text{ad}(x) \in \text{Der}(\mathfrak{g})$, for all x in $\mathfrak{g}$, and (b) the Jacobi identity holds in $\mathfrak{g}$. Derivations of the form $\text{ad}(x)$ are called *inner derivations*. Denote by $\text{ad}(\mathfrak{g})$ the subspace of $\text{Der}(\mathfrak{g})$ consisting of inner derivations. It is not difficult to prove that $\text{ad}(\mathfrak{g})$ is an ideal of $\text{Der}(\mathfrak{g})$. Elements in $\text{Der}(\mathfrak{g})$ which are not in $\text{ad}(\mathfrak{g})$ (if any) are called *outer derivations*.

We say that there exists a *linear representation* of $\mathfrak{g}$ in the vector space V if there exists a morphism of Lie algebras $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$. When V is a finite-dimensional vector space we can even ask for the existence of a morphism of Lie algebras $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(n)$ such that $\dim V = n$.

In particular, every Lie algebra can be represented in itself through its adjoint representation defined as follows,

$$\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}), \quad \text{where} \quad \text{ad}(x)(y) := [x, y]$$

for all $x, y \in \mathfrak{g}$. The fact that $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ is a representation is equivalent to request that the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfies the Jacobi identity, and it also is equivalent to state that $\text{ad}(x)$ is a derivation of $\mathfrak{g}$ for all $x \in \mathfrak{g}$.

Also, it is usual to define the ideal $Z(\mathfrak{g}) := \text{Ker}(\text{ad}) := \{x \in \mathfrak{g} \mid [x, y] = 0, \forall y \in \mathfrak{g}\}$ and it is called the *center* of $\mathfrak{g}$. Evidently, the adjoint representation $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ is injective if and only if $Z(\mathfrak{g}) = \{0\}$. From the previous examples we can conclude that $Z(\mathfrak{a}_n) = \mathfrak{a}_n$ for all $n \in \mathbb{N}$, whereas $Z(\mathfrak{so}(3)) = \{0\}$ and $Z(\mathfrak{gl}(n)) = \langle I_n \rangle$ for all $n \in \mathbb{N}$.

1.1. Solvable, nilpotent and semisimple Lie algebras. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h}, \mathfrak{m} \subseteq \mathfrak{g}$ be vector subspaces of $\mathfrak{g}$. Define,

$$[\mathfrak{h}, \mathfrak{m}] := \text{Span}_{\mathbb{F}}\{[x, y] \mid x \in \mathfrak{h}, y \in \mathfrak{m}\}.$$

In particular, if $\mathfrak{h}$ and $\mathfrak{m}$ are ideals of $\mathfrak{g}$, then $[\mathfrak{h}, \mathfrak{m}]$ is an ideal of $\mathfrak{g}$. We apply this result to the particular case $\mathfrak{h} = \mathfrak{m} = \mathfrak{g}$.

Given a Lie algebra $\mathfrak{g}$, it is defined the *derived series* of $\mathfrak{g}$ to be the series with terms

$$\mathfrak{g} \supset \mathfrak{g}^{(1)} \supset \dots \supset \mathfrak{g}^{(n)} \supset \dots$$

where $\mathfrak{g}^{(1)} := [\mathfrak{g}, \mathfrak{g}]$ is called the *first derived ideal* of $\mathfrak{g}$ and $\mathfrak{g}^{(n+1)} := [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}] = (\mathfrak{g}^{(n)})^{(1)}$ for all $n \geq 2$. This series is stabilized when $\mathfrak{g}$ is a finite-dimensional Lie algebra, that is, there exists $m \in \mathbb{N}$ such that $\mathfrak{g}^{(m+r)} = \mathfrak{g}^{(m)}$ for all $r \in \mathbb{N}$.

On the other hand, the *lower central series* is defined as follows:

$$\mathfrak{g} := \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \dots \supset \mathfrak{g}^n \supset \dots$$

where $\mathfrak{g}^2 := [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^{n+1} := [\mathfrak{g}, \mathfrak{g}^n]$ for all $n \geq 2$. This series is also stabilized when $\mathfrak{g}$ is a finite-dimensional Lie algebra, that is, there exists $m \in \mathbb{N}$ such that $\mathfrak{g}^{m+r} = \mathfrak{g}^m$ for all $r \in \mathbb{N}$.

The Lie algebra $\mathfrak{g}$ is said *g solvable* if there exists $m \in \mathbb{N}$ such that $\mathfrak{g}^{(m)} = \{0\}$. $\mathfrak{g}$ is said *nilpotent* if there exists $m \in \mathbb{N}$ such that $\mathfrak{g}^m = \{0\}$; the *nilpotency index* or *nilindex* of $\mathfrak{g}$ is the natural number k such that $\mathfrak{g}^r \neq \{0\}$ if $r < k$ and $\mathfrak{g}^k = \{0\}$.

The concepts of solvability and nilpotency imply that the respective series are stabilized in the trivial ideal $\{0\}$.

A direct calculation shows that $\mathfrak{g}^{(n)} \subset \mathfrak{g}^{n+1}$ for all $n \in \mathbb{N}$ and therefore, if $\mathfrak{g}$ is a nilpotent Lie algebra then $\mathfrak{g}$ is also a solvable Lie algebra. Usually, when we consider a solvable Lie algebra we are implicitly assuming that it is non-nilpotent.

The following Theorem is a well-known result:

Theorem 1.1 ([7], §1.3). *Let $\mathfrak{g}$ be a Lie algebra.*

- (1) *If $\mathfrak{h}$ and $\mathfrak{m}$ are solvable ideals of $\mathfrak{g}$, then $\mathfrak{h} + \mathfrak{m}$ is a solvable ideal of $\mathfrak{g}$.*
- (2) *If $\mathfrak{h}$ and $\mathfrak{m}$ are nilpotent ideals of $\mathfrak{g}$, then $\mathfrak{h} + \mathfrak{m}$ is a nilpotent ideal of $\mathfrak{g}$.*

From this result we have an immediate Corollary:

Corollary 1.2 ([7], §1.3). *Let $\mathfrak{g}$ be a Lie algebra. Then there exists a unique maximal solvable (respectively, nilpotent) ideal called the **radical** (respectively, **nilradical**) of $\mathfrak{g}$. This ideal is denoted by $\text{Rad}(\mathfrak{g})$ (respectively, $\text{NilRad}(\mathfrak{g})$).*

This Corollary suggests the following definitions, a Lie algebra $\mathfrak{g}$ is called *semisimple* if its radical is trivial, i.e., $\text{Rad}(\mathfrak{g}) = \{0\}$; whereas $\mathfrak{g}$ is called *simple* if $\dim \mathfrak{g} > 1$ and its only ideals are the trivial ones, i.e., if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ then either $\mathfrak{h} = \{0\}$ or $\mathfrak{h} = \mathfrak{g}$. We claim that every simple Lie algebra $\mathfrak{g}$ is also semisimple. First, observe that if $\mathfrak{g}$ is a simple Lie algebra, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Otherwise, if $[\mathfrak{g}, \mathfrak{g}] = \{0\}$, this would imply that $\mathfrak{g}$ is an abelian Lie algebra of dimension greater than one. Thus, any non-trivial subspace of $\mathfrak{g}$ would be a (non-trivial) ideal, which contradicts the fact that $\mathfrak{g}$ is a simple Lie algebra. Now, suppose that $\mathfrak{g}$ is not a semisimple Lie algebra, that is, its radical $\text{Rad}(\mathfrak{g})$ is a non-trivial solvable ideal. The simplicity of $\mathfrak{g}$ implies that $\mathfrak{g} = \text{Rad}(\mathfrak{g})$. On the other hand, due to the fact that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, it follows that $\mathfrak{g}^{(n)} = \mathfrak{g} \neq 0$ for all $n \in \mathbb{N}$, but this contradicts the fact that $\mathfrak{g} = \text{Rad}(\mathfrak{g})$. Therefore $\mathfrak{g}$ is semisimple.

The following Theorem is also a well-known result:

Theorem 1.3 ([7], §5.2). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over the field $\mathbb{F}$. The following statements are equivalent:*

- (1) *$\mathfrak{g}$ is semisimple, i.e., $\text{Rad}(\mathfrak{g}) = \{0\}$.*
- (2) *$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ where each $\mathfrak{g}_i$ ($i = 1, \dots, r$) is a simple ideal of $\mathfrak{g}$.*

Our work employs this characterization of semisimplicity. However, the literature offers several equivalent characterizations of semisimplicity. For instance, see [7], [4], [8], [12] and [14].

1.2. The Levi-Maltcev Theorem. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras over the field $\mathbb{F}$. Suppose that there is a representation of $\mathfrak{g}$ into $\mathfrak{h}$ *acting by derivations*, i.e., there is a Lie algebra morphism

$$\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{h}),$$

such that $\rho(x) \in \text{Der}(\mathfrak{h})$ for all $x \in \mathfrak{g}$. Now define a Lie bracket in $\mathfrak{g} \oplus \mathfrak{h}$ as follows:

$$[x + a, y + b] := [x, y]_{\mathfrak{g}} + \rho(x)(a) - \rho(y)(b) + [a, b]_{\mathfrak{h}},$$

for all $x, y \in \mathfrak{g}$ and for all $a, b \in \mathfrak{h}$. This new Lie algebra is called the *semidirect sum* of $\mathfrak{g}$ and $\mathfrak{h}$ and it is denoted by $\mathfrak{g} \ltimes \mathfrak{h}$. Evidently $\mathfrak{g}, \mathfrak{h} \subset \mathfrak{g} \ltimes \mathfrak{h}$, and observe that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g} \ltimes \mathfrak{h}$ whereas $\mathfrak{h}$ is an ideal of $\mathfrak{g} \ltimes \mathfrak{h}$. Furthermore, if $\rho(x) \equiv 0$ for all $x \in \mathfrak{g}$, then the resulting Lie algebra is called the *direct sum* of $\mathfrak{g}$ and $\mathfrak{h}$ and it is denoted by $\mathfrak{g} \oplus \mathfrak{h}$. Observe that the sum mentioned in the second statement of Theorem 1.3 corresponds precisely to a direct sum of Lie algebras.

Theorem 1.4 ([8], §3.9). *Let $\mathfrak{g}$ be a Lie algebra, then there exists a semisimple Lie subalgebra $\mathfrak{g}_L$ of $\mathfrak{g}$ such that*

$$\mathfrak{g} = \mathfrak{g}_L \ltimes \text{Rad}(\mathfrak{g}).$$

The subalgebra $\mathfrak{g}_L$ of $\mathfrak{g}$ is called a *Levi subalgebra* of $\mathfrak{g}$. Theorem 1.4 is known as the Levi's Theorem or the Levi-Maltcev Theorem.

The importance of the Levi-Maltcev Theorem lies in its consequential insight: to classify Lie algebras, it suffices to focus on two fundamental families –the semisimple Lie algebras and the solvable Lie algebras. Semisimple Lie algebras have been studied

by several mathematicians. Among these, the contributions of E. Cartan and W. Killing stand out prominently.

Cartan's classification of semisimple Lie algebras relies on two fundamental facts associated with the Lie algebra $\mathfrak{g}$: first, the existence of an invariant, non-degenerate, symmetric bilinear form on $\mathfrak{g}$, known as the *Cartan-Killing metric*; second, the existence of a special Lie subalgebra –called the *Cartan subalgebra*, having the property that the restriction of the Cartan-Killing metric to this Lie subalgebra is positive definite. The second fact allows for the study of so called *root systems* associated with the Cartan subalgebra from a geometric perspective.

1.3. Two extensions and the double extension process.

1.3.1. *Extending a Lie algebra by a derivation.* The first process to *extend* a Lie algebra comes from the construction of the semidirect sum. More specifically, let $\mathfrak{g}$ be a Lie algebra and let $D \in \text{Der}(\mathfrak{g})$ be a derivation of $\mathfrak{g}$. Consider the vector space defined by

$$\mathfrak{g}(D) := \langle D \rangle \oplus \mathfrak{g},$$

where $\langle D \rangle := \text{Span}_{\mathbb{F}}\{D\}$ and define its Lie algebra structure as the semidirect sum, i.e., $\mathfrak{g}(D) := \langle D \rangle \ltimes \mathfrak{g}$, where

$$[x + aD, y + bD] := [x, y]_{\mathfrak{g}} + aD(y) - bD(x),$$

for all $x, y \in \mathfrak{g}$, $a, b \in \mathbb{F}$. Clearly, $\mathfrak{g}$ is an ideal of codimension one in $\mathfrak{g}(D)$. Then, the following question arises naturally:

Under what conditions do there exist an ideal $\mathfrak{g}$ of codimension one in a Lie algebra $\mathfrak{h}$ and a derivation $D \in \text{Der}(\mathfrak{g})$ satisfying $\mathfrak{g} = \mathfrak{g}(D)$?

In other words, we want to determine the conditions for which a Lie algebra $\mathfrak{h}$ can be expressed as the image of a codimension-one ideal $\mathfrak{g}$ under a suitable derivation D . To completely answer this question, let us recall that the first derived ideal of a Lie algebra $\mathfrak{h}$ is denoted by $[\mathfrak{h}, \mathfrak{h}]$ and satisfies that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. First, suppose that $\dim[\mathfrak{h}, \mathfrak{h}] < \dim \mathfrak{h}$.

Let $\mathfrak{g}$ be a subspace of codimension one in $\mathfrak{h}$ such that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{g}$. Observe that $\mathfrak{g}$ is an ideal of $\mathfrak{h}$ since if $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$ then

$$[x, y] \in [\mathfrak{g}, \mathfrak{h}] \subset [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{g}.$$

Let $z \in \mathfrak{h}$ such that $z \notin [\mathfrak{h}, \mathfrak{h}]$ and $\mathfrak{h} = \langle z \rangle \oplus \mathfrak{g}$, we shall prove that $\mathfrak{h} \simeq \langle \text{ad}(z) \rangle \ltimes \mathfrak{g} = \mathfrak{g}(\text{ad}(z))$. For doing this, define $\varphi : \mathfrak{h} = \langle z \rangle \oplus \mathfrak{g} \rightarrow \langle \text{ad}(z) \rangle \oplus \mathfrak{g}$ in the following way, $z \mapsto \text{ad}(z)$ and $x \mapsto x$ for all $x \in \mathfrak{g}$. Evidently, $\varphi|_{\mathfrak{g}} = \text{Id}_{\mathfrak{g}}$ and therefore it is a morphism of Lie algebras. Finally observe that $\varphi([z, x]) = [z, x]$ if $x \in \mathfrak{g}$ since $\mathfrak{g}$ is an ideal, whereas $[\varphi(z), \varphi(x)] = [\text{ad}(z), x] = \text{ad}(z)(x) = [z, x]$ for all $x \in \mathfrak{g}$. Thus, φ is an isomorphism of Lie algebras.

On the other hand, if $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$ then it is not possible that there exists an ideal $\mathfrak{g}$ of codimension one in $\mathfrak{h}$ neither a derivation $D \in \text{Der}(\mathfrak{g})$ such that $\mathfrak{h} = \mathfrak{g}(D)$ since

$$[\mathfrak{g}(D), \mathfrak{g}(D)] = [\mathfrak{g} \oplus \langle D \rangle, \mathfrak{g} \oplus \langle D \rangle] \subset [\mathfrak{g}, \mathfrak{g}] + D(\mathfrak{g}) \subset \mathfrak{g} \neq \mathfrak{h}.$$

In summary, we have proved,

Theorem 1.5. *Let $\mathfrak{h}$ be a Lie algebra such that $[\mathfrak{h}, \mathfrak{h}] \neq \mathfrak{h}$, then there exist an ideal $\mathfrak{g}$ of codimension one in $\mathfrak{h}$ and a derivation $D \in \text{Der}(\mathfrak{g})$ such that*

$$\mathfrak{h} \simeq \mathfrak{g}(D) = \langle D \rangle \ltimes \mathfrak{g}.$$

As a consequence, we obtain the following result:

Corollary 1.6. *Let $\mathfrak{h}$ be a non-trivial solvable Lie algebra (respectively, nilpotent) Lie algebra. Then, there exists a solvable (respectively, nilpotent) ideal $\mathfrak{g}$ of codimension one, and a derivation (respectively, nilpotent derivation) $D \in \text{Der}(\mathfrak{g})$ such that*

$$\mathfrak{h} \simeq \mathfrak{g}(D) = \langle D \rangle \ltimes \mathfrak{g}.$$

Proof. Let $\mathfrak{h}$ be a solvable Lie algebra, if $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}^{(1)}$ then $\mathfrak{h}^{(n)} = \mathfrak{h}$ for all $n \in \mathbb{N}$. Since $\mathfrak{h}$ is solvable then there exists $m \in \mathbb{N}$ such that $\mathfrak{h} = \mathfrak{h}^{(m)} = \{0\}$, which is not possible. Hence $[\mathfrak{h}, \mathfrak{h}] \neq \mathfrak{h}$ and so, applying Theorem 1.5 we obtain the desired conclusion.

For the nilpotent case it remains to prove that the derivation D is also nilpotent. Let $\mathfrak{h}^r = \text{Span}_{\mathbb{F}}\{[x_1, [x_2, \dots, [x_{r-1}, x_r] \dots]]\}$ where $x_i \in \mathfrak{h}$, $1 \leq i \leq r$. If $\mathfrak{h}^n = 0$, then

$$[x_1, [x_2, \dots, [x_{n-1}, x_n] \dots]] = 0$$

for all $x_i \in \mathfrak{h}$. In particular, if $x_1 = \cdots = x_{n-1} = D$ and $x_n = x \in \mathfrak{g}$ it follows that

$$0 = [x_1, [x_2, \dots, [x_{n-1}, x_n] \dots]] = [D, [D, \dots, [D, x] \dots]] = D^{n-1}(x),$$

that is, D is nilpotent. □

We also can state the following result:

Theorem 1.7. *Let $\mathfrak{g}$ be a Lie algebra and let $D, E \in \text{Der}(\mathfrak{g})$ be derivations of $\mathfrak{g}$. Then $\mathfrak{g}(D) \simeq \mathfrak{g}(E)$ if and only if, there exist $a \in \mathbb{F} - \{0\}$, $v \in \mathfrak{g}$ and $\varphi \in \text{Aut}(\mathfrak{g})$ such that*

$$(1) \quad E = \frac{1}{a}(\varphi \circ D \circ \varphi^{-1} - \text{ad}(v)).$$

Proof. First, suppose that $\Phi : \mathfrak{g}(E) \rightarrow \mathfrak{g}(D)$ is an isomorphism of Lie algebras. Because of $[\mathfrak{g}(D), \mathfrak{g}(D)] \subseteq \mathfrak{g}$ and $[\mathfrak{g}(E), \mathfrak{g}(E)] \subseteq \mathfrak{g}$, Φ is given as follows: for all $x \in \mathfrak{g}$, $x \mapsto \varphi(x)$ where $\varphi \in \text{Aut}(\mathfrak{g})$, whereas $E \mapsto v + aD$ with $v \in \mathfrak{g}$, $a \in \mathbb{F} - \{0\}$. Since $\Phi : \mathfrak{g}(E) \rightarrow \mathfrak{g}(D)$ is a morphism of Lie algebras, $\Phi[D, x] = [\Phi(D), \Phi(x)]$ for all $x \in \mathfrak{g}$, then solving for E in this equation we obtain

$$E = \frac{1}{a}(\varphi \circ D \circ \varphi^{-1} - \text{ad}(v)).$$

On the other hand, suppose that there exist $a \in \mathbb{F} - \{0\}$, $v \in \mathfrak{g}$ and $\varphi \in \text{Aut}(\mathfrak{g})$ satisfying Equation (1). Define $\Phi : \mathfrak{g}(E) \rightarrow \mathfrak{g}(D)$ by $\Phi(x) = \varphi(x)$ for all $x \in \mathfrak{g}$ and $\Phi(E) = v + aD$. Clearly Φ is bijective. To show that Φ is a morphism of Lie algebras it is enough to verify that $\Phi[D, x] = [\Phi(D), \Phi(x)]$ for all $x \in \mathfrak{g}$, and this follows from Equation (1). □

1.3.2. Central Extensions. Given a Lie algebra, we want to construct a new Lie algebra having a non-trivial center. To do this, let $e \notin \mathfrak{g}$ be an element, and let $\langle e \rangle = \text{Span}_{\mathbb{F}}\{e\}$ be the vector space spanned by e . Consider now the direct sum of vector spaces $\mathfrak{g} \oplus \langle e \rangle$. We want to define a Lie bracket on this vector space in such a way that e becomes a *central element*; that is, $[x, e] = 0$ for all $x \in \mathfrak{g}$. It follows from this discussion that for all $x, y \in \mathfrak{g}$ one must have that $[x, y] \in \mathfrak{g} \oplus \langle e \rangle$, thus

$$[x, y] = [x, y]_{\mathfrak{g}} + \theta(x, y)e,$$

where $\theta : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{F}$ should be a bilinear function.

Observe that if $(\mathfrak{g} \oplus \langle e \rangle, [\cdot, \cdot])$ is a Lie algebra, the skew-symmetry of the bracket $[\cdot, \cdot]$ implies that θ should be skew-symmetric, i.e., $\theta(x, y) = -\theta(y, x)$ for all $x, y \in \mathfrak{g}$. On the other hand, notice that

$$[[x, y], z] = [[x, y]_{\mathfrak{g}} + \theta(x, y)e, z] = [[x, y]_{\mathfrak{g}}, z]_{\mathfrak{g}} + \theta([x, y]_{\mathfrak{g}}, z)e,$$

hence the Jacobi identity holds if and only if,

$$(2) \quad \theta([x, y]_{\mathfrak{g}}, z) + \theta([y, z]_{\mathfrak{g}}, x) + \theta([z, x]_{\mathfrak{g}}, y) = 0$$

for all $x, y, z \in \mathfrak{g}$.

Denote $(\mathfrak{g} \oplus \langle e \rangle, [\cdot, \cdot])$ by $\mathfrak{g}_{\theta}(e)$. This Lie algebra is called a *central extension* of $\mathfrak{g}$ by θ . The skew-symmetric bilinear functions $\theta : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{F}$ satisfying Equation (2) are called *2-cocycles* or *2-closed forms*.

In fact, we can prove the following result:

Theorem 1.8 ([6]). *Let $\mathfrak{g}$ be a Lie algebra and let $\alpha, \beta : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{F}$ be skew-symmetric bilinear functions satisfying Equation (2). Then $\mathfrak{g}_{\alpha}(e) \simeq \mathfrak{g}_{\beta}(e)$ if and only if, there exists $f \in \mathfrak{g}^*$ such that $\alpha(x, y) - \beta(x, y) = f([x, y])$ for all $x, y \in \mathfrak{g}$.*

Proof. Suppose that $\varphi : \mathfrak{g}_{\alpha}(e) \rightarrow \mathfrak{g}_{\beta}(e)$ is an isomorphism of Lie algebras. We know that φ maps the center of $\mathfrak{g}_{\alpha}(e)$ into the center of $\mathfrak{g}_{\beta}(e)$, then, it is enough to consider φ given as follows: for all $x \in \mathfrak{g}$, $\varphi(x) = x + f(x)e$ with $f \in \mathfrak{g}^*$ and $\varphi(e) = e$. Since $\varphi : \mathfrak{g}_{\alpha}(e) \rightarrow \mathfrak{g}_{\beta}(e)$ is a morphism of Lie algebras, then $\varphi[x, y] = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}$. It then follows that $\alpha(x, y) - \beta(x, y) = -f([x, y])$ for all $x, y \in \mathfrak{g}$.

Assume now that there exists $f \in \mathfrak{g}^*$ such that $\alpha(x, y) - \beta(x, y) = f([x, y])$ for all $x, y \in \mathfrak{g}$. Define $\varphi : \mathfrak{g}_{\alpha}(e) \rightarrow \mathfrak{g}_{\beta}(e)$ by $\varphi(x) = x + f(x)e$ for all $x \in \mathfrak{g}$, and $\varphi(e) = e$. Clearly, φ is bijective. To show that φ is a morphism of Lie algebras, it is enough to check that $\varphi[x, y] = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}$, and this follows from Equation (2). $\square$

The following example shows that we can begin with non-isomorphic Lie algebras, say $\mathfrak{g}$ and $\mathfrak{h}$, to perform a central extension on each one of them by some 2-cocycles α in $\mathfrak{g}$ and β in $\mathfrak{h}$, respectively, to obtain isomorphic Lie algebras, that is, $\mathfrak{g}_{\alpha}(e) \simeq \mathfrak{h}_{\beta}(e)$.

Example 1.9 (Non-isomorphic Lie algebras whose central extensions are isomorphic). First, consider the 3-dimensional abelian Lie algebra, $\mathfrak{a}_3 := \langle e_1, e_2, e_3 \rangle$. Let $\alpha : \mathfrak{a}_3 \times \mathfrak{a}_3 \rightarrow \mathbb{F}$ be the skew-symmetric bilinear 2-cocycle defined by $\alpha(e_1, e_2) = 1$, and $\alpha(e_2, e_3) = 0 = \alpha(e_3, e_1)$. A direct computation shows that $(\mathfrak{a}_3)_\alpha(e_4) = \langle e_1, e_2, e_3, e_4 \rangle$ is a central extension of $\mathfrak{a}_3$ by α and this central extension only has a non-trivial Lie bracket defined by

$$[e_1, e_2] = e_4.$$

On the other hand, consider the Lie algebra $\mathfrak{h}_3 = \langle e_1, e_2, e_3 \rangle$ defined by $[e_1, e_2] = e_3$. It is easy to check that $\mathfrak{a}_3$ and $\mathfrak{h}_3$ are non-isomorphic Lie algebras, if they would be isomorphic, then every bracket in $\mathfrak{h}_3$ would be zero. Define a trivial 2-cocycle $\beta : \mathfrak{h}_3 \times \mathfrak{h}_3 \rightarrow \mathbb{F}$ as $\beta(x, y) = 0$ for all $x, y \in \mathfrak{h}_3$, and construct the central extension of $\mathfrak{h}_3$ by β , $(\mathfrak{h}_3)_\beta(e_4) = \langle e_1, e_2, e_3, e_4 \rangle$. Since β is trivially defined as zero, the only non-trivial bracket in this central extension is given by

$$[e_1, e_2] = e_3.$$

Now, define $\Phi : (\mathfrak{a}_3)_\alpha(e_4) \rightarrow (\mathfrak{h}_3)_\beta(e_4)$ by $e_1 \mapsto e_1$, $e_2 \mapsto e_2$, $e_3 \mapsto e_4$ and $e_4 \mapsto e_3$. A straightforward calculation shows that Φ is an isomorphism of Lie algebras.

1.3.3. Double extensions. This process was introduced by V. Kac to construct finite-dimensional Lie algebras endowed with an *invariant metric* (see [9], §2, exercise 2.10). Let $\mathfrak{g}$ be a Lie algebra and let θ be a 2-cocycle. First, construct the central extension of $\mathfrak{g}$ by θ , i.e., $\mathfrak{g}_\theta(e)$. Then, consider a derivation $D \in \text{Der}(\mathfrak{g}_\theta(e))$ and construct the semidirect sum of $\langle D \rangle$ and $\mathfrak{g}_\theta(e)$ to obtain:

$$\mathfrak{g}(D, \theta) := \langle D \rangle \ltimes \mathfrak{g}_\theta(e)$$

the *double extension* of $\mathfrak{g}$ by the pair (D, θ) .

In view of the Example 1.9, it is clear from the definition that we can have non-isomorphic Lie algebras as $\mathfrak{g} = \mathfrak{a}_3$ and $\mathfrak{h} = \mathfrak{h}_3$, whose double extensions are isomorphic. In this case it is enough to extend $(\mathfrak{a}_3)_\alpha(e_4) = (\mathfrak{h}_3)_\beta(e_4)$ by any derivation D to obtain

$$\mathfrak{a}_3(D, \alpha) = \mathfrak{h}_3(D, \beta).$$

Thus, Example 1.9 demonstrates that the double extension process enables the construction of large families of Lie algebras. However, it does not serve as a useful tool for classifying these families.

1.4. Lie algebras that cannot be described as double extensions. We mention before that it is not possible to construct *all* Lie algebras through the process of a double extension. Thus, we show some examples of Lie algebras and families that cannot be obtained using this process. The first examples appear in dimension 3.

Example 1.10. Consider the Lie algebra $\mathfrak{sl}(2) = \langle H, E, F \rangle$ where the Lie brackets are defined by

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

Since $\mathfrak{sl}(2) = [\mathfrak{sl}(2), \mathfrak{sl}(2)]$, there does not exist an ideal $\mathfrak{g}$ of codimension one in $\mathfrak{sl}(2)$ neither a derivation $D \in \text{Der}(\mathfrak{g})$ such that

$$\mathfrak{sl}(2) = \mathfrak{g}(D).$$

In fact, one can prove that $\mathfrak{sl}(2)$ only has trivial ideals.

Example 1.11. Let $\mathfrak{so}(3) = \langle e_1, e_2, e_3 \rangle$ be the Lie algebra with brackets defined by

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.$$

As in the $\mathfrak{sl}(2)$ case, one can prove that $\mathfrak{so}(3)$ only has trivial ideals and hence, $\mathfrak{so}(3)$ cannot be obtained as a double extension of a Lie subalgebra of codimension one.

The Lie algebras $\mathfrak{sl}(2)$ and $\mathfrak{so}(3)$ are *simple* Lie algebras, i.e., they are Lie algebras having dimension greater than 1 and such that they only have trivial ideals. Therefore a simple Lie algebra cannot be described as a double extension. Moreover, since every *semisimple* Lie algebra is a sum of simple Lie algebras, one also obtains that a semisimple Lie algebra cannot be described as a double extension.

In the following example we present a Lie algebra that is neither simple nor semisimple and that it cannot be described as a double extension.

Example 1.12. Define $\mathfrak{h} = \mathfrak{sl}(2) \ltimes \mathbb{R}^2$ where $\mathfrak{sl}(2) = \langle H, E, F \rangle$, $\mathbb{R}^2 = \langle e_1, e_2 \rangle$ and the Lie brackets are defined by

$$\begin{aligned} [H, E] &= 2E, & [H, F] &= -2F, & [E, F] &= H, \\ [H, e_1] &= e_1, & [H, e_2] &= -e_2, & [E, e_1] &= 0, \\ [E, e_2] &= e_1, & [F, e_1] &= e_2, & [F, e_2] &= 0, & [e_1, e_2] &= 0. \end{aligned}$$

Suppose that $\mathfrak{h}$ admits an ideal $\mathfrak{g} \subset \mathfrak{h}$ of codimension one, and that there exists a derivation $D \in \text{Der}(\mathfrak{g})$ such that $\mathfrak{h} = \mathfrak{g}(D)$. Let $x \in \mathfrak{g} \subset \mathfrak{h}$, then

$$x = aH + bE + cF + ge_1 + he_2,$$

since $\mathfrak{g}$ is an ideal, then

$$\mathfrak{g} \ni [x, F] = -2aF + bH - ge_2,$$

and one also has that

$$\mathfrak{g} \ni [[x, F], E] = 2aH + 2bE.$$

If $a \neq 0$ then, multiplying again by E , we obtain that $E \in \mathfrak{g}$ and this implies that $H, E, F \in \mathfrak{g}$, i.e., $\mathfrak{sl}(2) \subset \mathfrak{g}$. If $a = 0$ and $b \neq 0$, then $E \in \mathfrak{g}$ and again one has that $H, E, F \in \mathfrak{g}$, i.e., $\mathfrak{sl}(2) \subset \mathfrak{g}$. Analogously, if $a = b = 0$, then $F \in \mathfrak{g}$ and hence $H, E, F \in \mathfrak{g}$, i.e., $\mathfrak{sl}(2) \subset \mathfrak{g}$. Finally, observe that it is not possible to have $a = b = c = 0$ for all $x \in \mathfrak{g}$, since this would imply that the dimension of $\mathfrak{g}$ is at most 2, but we know that $\mathfrak{h}$ is a 5-dimensional Lie algebra and $\mathfrak{g}$ is an ideal of codimension one in $\mathfrak{h}$.

We have proved that if $\mathfrak{g} \subset \mathfrak{h}$ is an ideal of codimension one, then $\mathfrak{sl}(2) \subset \mathfrak{g}$. Since $\mathfrak{g}$ is a 4-dimensional ideal, then there exists a vector $u \in \mathbb{R}^2 - \{0\}$, say $u = ae_1 + be_2$, such that

$$\mathfrak{g} = \mathfrak{sl}(2) \oplus \langle u \rangle.$$

The last equality implies that $[H, u] \in \langle u \rangle$, i.e., $[H, u] = \lambda u$. However

$$\lambda(ae_1 + be_2) = [H, ae_1 + be_2] = ae_1 - be_2$$

if and only if

$$\lambda a = a, \quad \text{and} \quad \lambda b = -b.$$

If $a, b \neq 0$ we obtain a contradiction since λ must be equal to 1 and -1. Thus, with no loss of generality, we can suppose that $a \neq 0$ and $b = 0$, that is, $u = ae_1$, and since $a \neq 0$, then we can *normalize* to assume that

$$u = e_1.$$

Observe that $[F, e_1] \in \langle e_1 \rangle$ leads to a contradiction, since $[F, e_1] = e_2 \notin \langle e_1 \rangle$. An analogous argument shows that the case $a = 0$ and $b \neq 0$ also produces a contradiction.

We can conclude that it is not possible to describe $\mathfrak{h}$ as a double extension of an ideal $\mathfrak{g}$ of codimension one and a derivation $D \in \text{Der}(\mathfrak{g})$.

The next example provides a 5-dimensional *solvable* Lie algebra that cannot be described as a double extension of a 3-dimensional Lie algebra.

Example 1.13. Consider $\mathfrak{a}_3 = \langle e_1, e_2, e_3 \rangle$ and let $S, T \in \text{Der}(\mathfrak{a}_3)$ be derivations such that with respect to the basis $\{e_1, e_2, e_3\}$ have the following matrix representations,

$$[S] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad [T] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Let $e_4 := S$ and $e_5 := T$ and define $\mathfrak{h} = \langle e_4, e_5 \rangle \ltimes \mathfrak{a}_3$. If $\mathfrak{h} = \mathfrak{g}(D)$ with $\mathfrak{g} \subset \mathfrak{h}$ being an ideal of codimension one in $\mathfrak{h}$, then $Z(\mathfrak{g}) \neq \{0\}$. However, it can be proved that if $\mathfrak{g}$ is an ideal of codimension one in $\mathfrak{h}$, then $Z(\mathfrak{g}) = \{0\}$. Hence it is not possible to describe $\mathfrak{h}$ as double extension of a 3-dimensional Lie algebra.

In the last example observe that $\mathfrak{a}_2 = \langle e_4, e_5 \rangle$ and thus, the Lie algebra structure of Example 1.13 is $\mathfrak{a}_2 \ltimes \mathfrak{a}_3$. A similar analysis shows that the Lie algebras of the form $\mathfrak{a}_{2n} \ltimes \mathfrak{a}_{2n+1}$, constructed in a similar way as before, cannot be obtained as double extensions.

Summarizing, the following families of Lie algebras cannot be described as double extensions: simple Lie algebras, semisimple Lie algebras and the family of Lie algebras $\mathfrak{a}_{2n} \ltimes \mathfrak{a}_{2n+1}$.

1.5. Nilpotent Lie algebras. While constructing every Lie algebra using the double extension process is not feasible, we can pose the following question:

Is there a family of Lie algebras where each element can be obtained through the double extension process from another Lie algebra within the same family?

The answer is affirmative. The family of nilpotent Lie algebras satisfies these conditions.

To prove this, first observe that if $\mathfrak{h}$ is a nilpotent Lie algebra, then $\mathfrak{h} \neq [\mathfrak{h}, \mathfrak{h}]$, hence, as a consequence of Corollary 1.6, we can conclude that there exists an ideal $\mathfrak{g} \subset \mathfrak{h}$ of codimension one and a derivation $D \in \text{Der}(\mathfrak{g})$ such that

$$\mathfrak{h} = \mathfrak{g}(D) = \langle D \rangle \ltimes \mathfrak{g}.$$

Another immediate consequences are that $\mathfrak{g}$ is nilpotent and D is a nilpotent derivation of $\mathfrak{g}$. Then, we only need to prove that $\mathfrak{g}$ is a central extension of some Lie algebra. To do this, we have to use the fact that the nilpotency of $\mathfrak{g}$ implies that $Z(\mathfrak{g}) \neq \{0\}$.

Suppose that $\mathfrak{g}$ is a non-abelian nilpotent Lie algebra, then $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}] \neq \{0\}$. To see this, assume that $\mathfrak{g}^m = \{0\}$ and that $\mathfrak{g}^j \neq \{0\}$ for $1 \leq j < m$. Since $\{0\} = \mathfrak{g}^m = [\mathfrak{g}, \mathfrak{g}^{m-1}]$, we have that $\{0\} \neq \mathfrak{g}^{m-1} \subset Z(\mathfrak{g})$ and $\mathfrak{g}^{m-1} \subset [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^2$. Therefore $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}] \neq \{0\}$.

Let $e \in Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ such that $e \neq 0$. Since $e \in [\mathfrak{g}, \mathfrak{g}]$, it follows that there exist skew-symmetric bilinear maps $\theta : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{F}$ and $F : \mathfrak{g} \times \mathfrak{g} \rightarrow [\mathfrak{g}, \mathfrak{g}]$ such that $[x, y] = \theta(x, y)e + F(x, y)$ for all $x, y \in \mathfrak{g}$.

Choosing a basis of $[\mathfrak{g}, \mathfrak{g}]$, namely, $\{e, x_1, \dots, x_r\}$, we have that $F(x, y)$ can be written as a linear combination of $\{x_1, \dots, x_r\}$. That is, $F(x, y) = c_1(x, y)x_1 + \dots + c_r(x, y)x_r$. Thus

$$[x, y] = \theta(x, y)e + c_1(x, y)x_1 + \dots + c_r(x, y)x_r$$

for all $x, y \in \mathfrak{g}$. Moreover, if we extend $\{e, x_1, \dots, x_r\}$ to a basis of $\mathfrak{g}$, say $\{e, x_1, \dots, x_r, y_1, \dots, y_s\}$, we can state that $\mathfrak{n} = \langle x_1, \dots, x_r, y_1, \dots, y_s \rangle$ is a nilpotent Lie algebra since $e \in Z(\mathfrak{g})$. The

Lie algebra structure of $\mathfrak{n}$ is defined by the multiplication map F . Clearly, the skew-symmetric bilinear map θ is a 2-cocycle of the Lie algebra $\mathfrak{n}$. Therefore, $\mathfrak{g}$ is a central extension of the Lie subalgebra $\mathfrak{n}$ by θ ,

$$\mathfrak{g} = \mathfrak{n}_\theta(e).$$

If the Lie algebra $\mathfrak{g}$ is nilpotent and abelian, then $\mathfrak{g} = \mathfrak{a}_r$ where $r = \dim \mathfrak{g}$, thus, $\mathfrak{g} = \mathfrak{a}_r = \mathfrak{a}_{r-1} \oplus \mathfrak{a}_1$ and hence it is a central extension of $\mathfrak{a}_{r-1}$.

We have proved the following:

Theorem 1.14. *Let $\mathfrak{h}$ be a nilpotent Lie algebra, then there exists a nilpotent Lie algebra $\mathfrak{n}$, a 2-cocycle θ of $\mathfrak{n}$ and a nilpotent derivation $D \in \text{Der}(\mathfrak{n}_\theta(e))$ such that*

$$\mathfrak{h} = \mathfrak{n}(D, \theta) = \langle D \rangle \ltimes \mathfrak{n}_\theta(e).$$

2. CONSTRUCTION AND CLASSIFICATION OF LOW-DIMENSIONAL LIE ALGEBRAS

It follows from Theorem 1.14 that we shall focus our attention into the classification of low-dimensional solvable Lie algebras. Specifically, we want to show that most of them can be obtained through the double extension process. Recall that two Lie algebras are isomorphic if there exists a bijective morphism of Lie algebras $\varphi : \mathfrak{g} \rightarrow \mathfrak{h}$. In particular we shall use the following result.

Theorem 2.1. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be isomorphic Lie algebras and $\varphi : \mathfrak{g} \rightarrow \mathfrak{h}$ be a isomorphism of Lie algebras. Then:*

- (1) $\varphi(Z(\mathfrak{g})) = Z(\mathfrak{h})$.
- (2) $\varphi([\mathfrak{g}, \mathfrak{g}]) = [\varphi(\mathfrak{g}), \varphi(\mathfrak{g})] = [\mathfrak{h}, \mathfrak{h}]$.

Proof. To prove the first statement, suppose that $\varphi : \mathfrak{g} \rightarrow \mathfrak{h}$ is an isomorphism between $\mathfrak{g}$ and $\mathfrak{h}$. Let $x \in Z(\mathfrak{g})$, and let $h \in \mathfrak{h}$, since φ is an isomorphism there exists $y \in \mathfrak{g}$ such that $h = \varphi(y)$, thus:

$$[\varphi(x), h] = [\varphi(x), \varphi(y)] = \varphi([x, y]) = \varphi(0) = 0,$$

hence $\varphi(x) \in Z(\mathfrak{h})$ and $\varphi(Z(\mathfrak{g})) \subset Z(\mathfrak{h})$.

Suppose now that $h \in Z(\mathfrak{h})$ and that $h = \varphi(x)$, with $x \in \mathfrak{g}$. Let $y \in \mathfrak{g}$, then

$$\varphi([x, y]) = [\varphi(x), \varphi(y)] = [h, \varphi(y)] = 0,$$

since φ is injective, we conclude that $[x, y] = 0$, therefore $x \in Z(\mathfrak{g})$, i.e., $Z(\mathfrak{h}) \subset \varphi(Z(\mathfrak{g}))$.

To prove the second statement it is enough to observe that $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}$, where $\varphi(\mathfrak{g}) = \mathfrak{h}$. $\square$

In other words, Theorem 2.1 implies that there exists some ideals that are *preserved* under isomorphisms. This fact establishes *necessary* but not sufficient conditions so that two Lie algebras are isomorphic.

2.1. Dimensions 1 and 2.

2.1.1. 1-dimensional Lie algebras. Suppose that $\mathfrak{g}$ and $\mathfrak{h}$ are 1-dimensional Lie algebras. Since the Lie bracket is skew-symmetric, we can conclude that $[\cdot, \cdot]_{\mathfrak{g}} \equiv 0 \equiv [\cdot, \cdot]_{\mathfrak{h}}$. Let $\mathfrak{g} = \langle e \rangle$ and $\mathfrak{h} = \langle f \rangle$, then $\varphi : \mathfrak{g} \rightarrow \mathfrak{h}$ defined by $\varphi(e) = f$ is clearly an isomorphism of Lie algebra, thus, *up to isomorphism*, there is only one 1-dimensional Lie algebra.

2.1.2. 2-dimensional Lie algebras. Let $\mathfrak{g}$ be a 2-dimensional Lie algebra and suppose that $\mathfrak{g} = \langle e_1, e_2 \rangle$. Since $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is a skew-symmetric bilinear function, it is defined by its values in the pair of basic elements, however, it is clear that $[e_1, e_1] = [e_2, e_2] = 0$, and hence the Lie bracket is completely determined by its value in the pair (e_1, e_2) . Furthermore, since $\{e_1, e_2\}$ is a basis, we must have that

$$[e_1, e_2] = ae_1 + be_2 =: u,$$

for some $a, b \in \mathbb{F}$. In other words $[\mathfrak{g}, \mathfrak{g}] = \langle u \rangle$, in particular $\dim[\mathfrak{g}, \mathfrak{g}] \leq 1$, and $\dim[\mathfrak{g}, \mathfrak{g}] = 1$ if and only if $u \neq 0$.

If $[\mathfrak{g}, \mathfrak{g}] = \{0\}$, then $[e_1, e_2] = 0$ and $\mathfrak{g} = \mathfrak{a}_2$.

If $[\mathfrak{g}, \mathfrak{g}] = \langle u \rangle \neq \{0\}$, then fix the vector u as the second basic element if $b \neq 0$, i.e., $\mathfrak{g} = \langle e_1, u = ae_1 + be_2 \rangle$, so we have that $[e_1, u] = bu$. Thus we can state that there exists a basis $\{f_1, f_2\}$ of $\mathfrak{g}$ such that $[f_1, f_2] = f_2$. Indeed, it is enough to define $f_1 = (1/b)e_1$ and $f_2 = u$. The case $a \neq 0$ is analogous.

We have proved

Theorem 2.2. *Let $\mathfrak{g}$ be a 2-dimensional Lie algebra, then $\mathfrak{g}$ is isomorphic to:*

- (1) $\mathfrak{a}_2$ if and only if $[\mathfrak{g}, \mathfrak{g}] = \{0\}$.
- (2) $\mathfrak{af}(2) = \langle e_1, e_2 \rangle$ and $[e_1, e_2] = e_2$ if and only if $[\mathfrak{g}, \mathfrak{g}] \neq \{0\}$.

Now we show that these solvable Lie algebras can be obtained by some extension process mentioned in the previous section. Let us start with the 1-dimensional abelian Lie algebra, $\mathfrak{a}_1$. It is clear that any skew-symmetric bilinear function $\theta : \mathfrak{a}_1 \times \mathfrak{a}_1 \rightarrow \mathbb{F}$ is trivial, i.e., $\theta \equiv 0$, thus $(\mathfrak{a}_1)_\theta(e_2)$ is the 2-dimensional abelian Lie algebra. In other words, the central extension of $\mathfrak{a}_1$ by the only non-trivial 2-cocycle that can be defined, produces the 2-dimensional abelian Lie algebra.

On the other hand, one has that $\text{Der}(\mathfrak{a}_1) = \text{End}(\mathfrak{a}_1)$. Consider $D : \mathfrak{a}_1 \rightarrow \mathfrak{a}_1$ defined by $D(e_1) = e_1$ and let $\mathfrak{a}_1(D) := \langle D \rangle \ltimes \mathfrak{a}_1$, the only non-trivial bracket is given by

$$[D, e_1] = D(e_1) = e_1.$$

Evidently, this Lie algebra is isomorphic to the Lie algebra $\mathfrak{af}(2)$.

This discussion allows to state that

Theorem 2.3. *Every 2-dimensional Lie algebra can be obtained either as a central extension of $\mathfrak{a}_1$ or as an extension of $\mathfrak{a}_1$ by a derivation.*

2.2. Dimension 3. Assume that $\mathfrak{g}$ is a 3-dimensional Lie algebra and that $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$ satisfies that $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$. Clearly this implies that $\dim[\mathfrak{g}, \mathfrak{g}] = 1, 2$. We show that in any case $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{a}_2$ and hence it always exists an abelian ideal of codimension one in $\mathfrak{g}$.

Suppose that $\dim[\mathfrak{g}, \mathfrak{g}] = 1$. We know that the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is defined by its values in the pair of basic elements, say (e_1, e_2) , (e_1, e_3) and (e_2, e_3) . Since the rank of this bilinear function is one, then its kernel is 2-dimensional, so with no loss of generality we can suppose that

$$[e_1, e_2] = [e_1, e_3] = 0 \quad \text{and} \quad [e_2, e_3] \neq 0,$$

this implies that $[e_2, e_3] = ae_1 + be_2 + ce_3$, with $a, b, c \in \mathbb{F}$. If $b \neq 0$ we can replace e_2 by $u = ae_1 + be_2 + ce_3$ to obtain

$$[e_1, u] = [e_1, e_3] = 0, \quad \text{and} \quad [u, e_3] = b[e_2, e_3] = bu \neq 0.$$

Certainly, we can choose a multiple of u , that we call again e_2 , in such a way that

$$[e_1, e_2] = [e_1, e_3] = 0 \quad \text{and} \quad [e_2, e_3] = e_2.$$

If $c \neq 0$, an analogous argument shows that one can choose a basis of $\mathfrak{g}$ such that:

$$[e_1, e_2] = [e_1, e_3] = 0 \quad \text{and} \quad [e_2, e_3] = e_2.$$

Finally, if $b = c = 0$ we can choose a multiple of e_1 in order to

$$[e_1, e_2] = [e_1, e_3] = 0 \quad \text{and} \quad [e_2, e_3] = e_1.$$

In any case, it is clear that $[\mathfrak{g}, \mathfrak{g}]$ is contained in the abelian Lie algebra spanned by the basic elements e_1 and e_2 .

Suppose that $\dim[\mathfrak{g}, \mathfrak{g}] = 2$. Let us say then that $[\mathfrak{g}, \mathfrak{g}] = \langle e_2, e_3 \rangle$ and that $[e_2, e_3] = ae_2 + be_3$ with $a, b \in \mathbb{F}$. Observe that $\text{ad}(e_2) : \mathfrak{g} \rightarrow \mathfrak{g}$ is a linear transformation. Then we can consider its restriction to $[\mathfrak{g}, \mathfrak{g}]$ and obtain its matrix representation, namely $\text{ad}(e_2)|_{[\mathfrak{g}, \mathfrak{g}]} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$. However, since $e_2 \in [\mathfrak{g}, \mathfrak{g}]$ the trace of $\text{ad}(e_2)$ has to be zero, thus, $b = 0$. An analogous argument shows that $a = 0$ and hence

$$[e_2, e_3] = 0.$$

Obviously, we can extend this basis with a vector $e_1 \in \mathfrak{g}$ and such that $e_1 \notin [\mathfrak{g}, \mathfrak{g}]$, therefore

$$[e_1, e_2] = ae_2 + ce_3, \quad [e_1, e_3] = be_2 + de_3, \quad [e_2, e_3] = 0$$

for some a, b, c and $d \in \mathbb{F}$. Changing the order of the basis vectors of $\mathfrak{g}$ in the case $\dim[\mathfrak{g}, \mathfrak{g}] = 1$, we have proved

Theorem 2.4. *Let $\mathfrak{g}$ be a 3-dimensional Lie algebra such that $\mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}]$, then there exists a basis of $\mathfrak{g}$, $\{e_1, e_2, e_3\}$, such that*

- (1) $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{a}_2 := \langle e_2, e_3 \rangle$,
- (2) $\text{ad}(e_1)|_{\mathfrak{a}_2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Der}(\mathfrak{a}_2)$, where $a, b, c, d \in \mathbb{F}$.

Corollary 2.5. *Every 3-dimensional Lie algebra $\mathfrak{g}$ such that $\mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}]$ can be obtained as a double extension of $\mathfrak{a}_1$.*

Proof. If $\mathfrak{g}$ satisfies the hypothesis, it follows from Theorem 2.4 that $\mathfrak{g}$ is obtained as an extension of $\mathfrak{a}_2$ by a derivation, that is,

$$\mathfrak{g} = \langle \text{ad}(e_1)|_{\mathfrak{a}_2} \rangle \ltimes \mathfrak{a}_2.$$

To finish the proof, just observe that $\mathfrak{a}_2 = (\mathfrak{a}_1)_\theta(e_2)$ is a central extension of $\mathfrak{a}_1$ by the trivial 2-cocycle $\theta \equiv 0$. $\square$

Observe that we have described every 3-dimensional Lie algebra satisfying $\mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}]$ as an extension of $\mathfrak{a}_2$ by a derivation $\mathfrak{a}_2(D) = \langle D \rangle \ltimes \mathfrak{a}_2$. Then the fact that $\text{ad}(x) = 0$ for all $x \in \mathfrak{a}_2$ together with Theorem 1.7 imply that

$$\mathfrak{a}_2(D) \simeq \mathfrak{a}_2(E), \quad \text{if and only if} \quad E = \frac{1}{a}(\varphi^{-1} \circ D \circ \varphi)$$

where $a \in \mathbb{F} - \{0\}$ and $\varphi \in \text{Aut}(\mathfrak{a}_2)$. Since $\mathfrak{a}_2$ is abelian we have that $\text{Aut}(\mathfrak{a}_2)$ is $\text{GL}(2)$ the isomorphism group of $\mathbb{F}^2$.

A careful analysis shows that:

Theorem 2.6. *Let $\mathfrak{g}$ be a 3-dimensional Lie algebra such that $\mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}]$. Preserving the notation of Theorem 2.4, it follows that $\mathfrak{g}$ is isomorphic to exactly only one Lie algebra of the following list:*

- (1) $\mathfrak{p}(\mathbb{F}), \text{ad}(e_1)|_{\mathfrak{a}_2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$
- (2) $\mathfrak{q}_\lambda(\mathbb{F}), \text{ad}(e_1)|_{\mathfrak{a}_2} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, 0 < |\lambda| \leq 1.$
- (3) $\mathfrak{q}_\lambda^1(\mathbb{R}), \text{ad}(e_1)|_{\mathfrak{a}_2} = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}, \lambda \in \mathbb{R}.$
- (4) $\mathfrak{q}_0(\mathbb{F}), \text{ad}(e_1)|_{\mathfrak{a}_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$
- (5) $\mathfrak{h}(\mathbb{F}), \text{ad}(e_1)|_{\mathfrak{a}_2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$
- (6) $\mathfrak{a}_3(\mathbb{F}), \text{ad}(e_1)|_{\mathfrak{a}_2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$

We also can state the following:

Theorem 2.7. *Let $\mathfrak{g}$ be a 3-dimensional real Lie algebra such that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Then $\mathfrak{g}$ is isomorphic to:*

- (1) $\mathfrak{sl}(2)$, or
- (2) $\mathfrak{so}(3) (\simeq \mathfrak{su}(2))$.

2.3. Dimensions 4 and 5. First, we need a definition: a finite dimensional Lie algebra is said *decomposable* if it can be written as a direct sum of non-trivial ideals. Otherwise, we say that $\mathfrak{g}$ is *indecomposable*. In general, classifying Lie algebras is a challenging task, and it becomes intricate even in low dimensions. The literature contains various classifications for low-dimensional Lie algebras. L. Boza et al [5] provide an extensive list of references related to the development of Lie algebra classification. Our focus in this section is to show that most of the 4- and 5-dimensional indecomposable Lie algebras over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ can be described as double extensions of Lie subalgebras of codimension 2.

Theorem 1.14 establishes that every nilpotent Lie algebra can be obtained as a double extension from another nilpotent Lie subalgebra having two dimensions less. Therefore, we narrow our attention to 4- and 5-dimensional indecomposable solvable Lie algebras and determine which among them can be described using the double extension process. To achieve this, we follow the work of L. Šnobl and P. Winternitz [15]. As far as we know, thier work provides the most recent classification of low-dimensional indecomposable Lie algebras.

From this moment, $\mathfrak{s}_{n,r}$ denotes an indecomposable (non-nilpotent) solvable Lie algebra, where the first subindex indicates the dimension ($n = 4, 5$), whereas the second one is used for enumeration. For example, $\mathfrak{s}_{4,10}$ stands for the tenth 4-dimensional indecomposable solvable Lie algebra appearing in the classification. Now, for the 4 and 5-dimensional Lie algebras presented in the classification of L. Šnobl and P. Winternitz, we use the basis $\{e_1, \dots, e_n\}$ ($n = 4, 5$) given there to determine (when it is possible) the double extension data: the Lie subalgebra $\mathfrak{g}$ of codimension 2 (as it is presented in the classification of low-dimensional Lie algebras that we are following), the 2-cocycle θ and the derivation D of the central extension of $\mathfrak{g}$ by θ . The derivation D is expressed

by its matrix representation in the fixed basis. If the 2-cocycle θ is non-trivial, we use the following notation: letting $n = 4, 5$, for some $i, j \in \{1, 2, \dots, n\}$ with $i < j$, θ_{ij} stands for $\theta(e_i, e_j) = 1$ and $\theta(e_k, e_\ell) = 0$ whenever $(k, \ell) \neq (i, j)$.

In dimension 4, up to isomorphism, we have 13 classes of real indecomposable Lie algebras, one of them corresponds to a nilpotent Lie algebra that clearly can be described as a double extension of a nilpotent Lie subalgebra of codimension 2, whereas the 12 remaining classes correspond to solvable Lie algebras, and in some cases, the representatives are families of Lie algebras that depend on parameters. From the 12 non-isomorphic classes of solvable Lie algebras, there is only one that cannot be expressed as a double extension, namely, the Lie algebra $\mathfrak{s}_{4,12}$. The list of the non-isomorphic 4-dimensional indecomposable solvable Lie algebras that can be described as a double extension is the following:

$\mathfrak{s}_{4,1}$

$$[e_4, e_2] = e_1, [e_4, e_3] = e_3$$

$$\text{Double extension data: } \mathfrak{a}_2 = \langle e_1, e_2 \rangle, \theta = 0, \text{ad}(e_4) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\mathfrak{s}_{4,2}$

$$[e_4, e_1] = e_1, [e_4, e_2] = e_1 + e_2, [e_4, e_3] = e_2 + e_3$$

$$\text{Double extension data: } \mathfrak{a}_2 = \langle e_1, e_2 \rangle, \theta = 0, \text{ad}(e_4) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\mathfrak{s}_{4,3}$

$$[e_4, e_1] = e_1, [e_4, e_2] = ae_2, [e_4, e_3] = be_3, \text{ where } 0 < |b| \leq |a| \leq 1, (a, b) \neq (-1, 1)$$

$$\text{Double extension data: } \mathfrak{a}_2 = \langle e_1, e_2 \rangle, \theta = 0, \text{ad}(e_4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}.$$

$\mathfrak{s}_{4,4}$

$$[e_4, e_1] = e_1, [e_4, e_2] = e_1 + e_2, [e_4, e_3] = ae_3 \text{ where } a \neq 0$$

$$\text{Double extension data: } \mathfrak{a}_2 = \langle e_1, e_2 \rangle, \theta = 0, \text{ad}(e_4) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}.$$

$\mathfrak{s}_{4,5}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{4,3}$)

$$[e_4, e_1] = \alpha e_1, [e_4, e_2] = \beta e_2 - e_3, [e_4, e_3] = e_2 + \beta e_3 \text{ where } \alpha > 0$$

$$\text{Double extension data: } \mathfrak{a}_2 = \langle e_1, e_2 \rangle, \theta = 0, \text{ad}(e_4) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 1 \\ 0 & -1 & \beta \end{pmatrix}.$$

$\mathfrak{s}_{4,6}$

$$[e_2, e_3] = e_1, [e_4, e_2] = e_2, [e_4, e_3] = -e_3$$

Double extension data: $\mathfrak{a}_2 = \langle e_2, e_3 \rangle, \theta_{23} = 1, \text{ad}(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

$\mathfrak{s}_{4,7}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{4,6}$)

$$[e_2, e_3] = e_1, [e_4, e_2] = -e_3, [e_4, e_3] = e_2$$

Double extension data: $\mathfrak{a}_2 = \langle e_2, e_3 \rangle, \theta_{23} = 1, \text{ad}(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$.

$\mathfrak{s}_{4,8}$

$$[e_2, e_3] = e_1, [e_4, e_1] = (1 + a)e_1, [e_4, e_2] = e_2, [e_4, e_3] = ae_3,$$

over the field $\mathbb{F} = \mathbb{C} : 0 < |a| \leq 1$, if $|a| = 1$ then $\arg(a) < \pi$;

over the field $\mathbb{F} = \mathbb{R} : -1 < a \leq 1, a \neq 0$.

Double extension data: $\mathfrak{a}_2 = \langle e_2, e_3 \rangle, \theta_{23} = 1, \text{ad}(e_4) = \begin{pmatrix} 1+a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

$\mathfrak{s}_{4,9}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{4,8}$)

$$[e_2, e_3] = e_1, [e_4, e_1] = 2\alpha e_1, [e_4, e_2] = \alpha e_2 - e_3, [e_4, e_3] = e_2 + \alpha e_3 \text{ where } \alpha > 0.$$

Double extension data: $\mathfrak{a}_2 = \langle e_2, e_3 \rangle, \theta_{23} = 1, \text{ad}(e_4) = \begin{pmatrix} 2\alpha & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & -1 & \alpha \end{pmatrix}$.

$\mathfrak{s}_{4,10}$

$$[e_2, e_3] = e_1, [e_4, e_1] = 2e_1, [e_4, e_2] = e_2, [e_4, e_3] = e_2 + e_3$$

Double extension data: $\mathfrak{a}_2 = \langle e_2, e_3 \rangle, \theta_{23} = 1, \text{ad}(e_4) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

$\mathfrak{s}_{4,11}$

$$[e_2, e_3] = e_1, [e_4, e_1] = e_1, [e_4, e_2] = e_2$$

Double extension data: $\mathfrak{a}_2 = \langle e_2, e_3 \rangle, \theta_{23} = 1, \text{ad}(e_4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$\mathfrak{s}_{4,12}$ (decomposable over the field $\mathbb{C}$)

$$[e_3, e_1] = e_1, [e_3, e_2] = e_2, [e_4, e_1] = -e_2, [e_4, e_2] = e_1$$

Double extension data: $-$.

On the other hand, up to isomorphism, there are 52 classes of real 5-dimensional Lie algebras, from which 6 classes correspond to indecomposable nilpotent Lie algebras, 45 classes correspond to indecomposable solvable Lie algebras and the remaining class corresponds to the decomposable Lie algebra $\mathfrak{sl}(2) \ltimes \mathbb{R}^2$. As in dimension 4, some of the representatives are families of Lie algebras depending on parameters. From the 45 indecomposable solvable Lie algebras there are just two of them that cannot be described as a double extension, namely $\mathfrak{s}_{5,42}$ and $\mathfrak{s}_{5,43}$. In fact, observe that $\mathfrak{s}_{5,43}$ with parameters $\alpha = \beta = 1$ is the Lie algebra $\mathfrak{a}_2 \ltimes \mathfrak{a}_3$ discussed in Example 1.13. On the other

hand, recall that in Example 1.12 we discussed that $\mathfrak{sl}(2) \ltimes \mathbb{R}^2$ cannot be described as a double extension. The list of the non-isomorphic 5-dimensional indecomposable solvable Lie algebras and its description (when it is possible) as a double extension is the following:

$\mathfrak{s}_{5,1}$

$$[e_5, e_2] = e_1, [e_5, e_3] = e_2, [e_5, e_4] = e_4.$$

$$\text{Double extension data: } \mathfrak{a}_3 = \langle e_1, e_2, e_3 \rangle, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathfrak{s}_{5,2}$

$$[e_5, e_2] = e_1, [e_5, e_3] = e_3, [e_5, e_4] = e_3 + e_4.$$

$$\text{Double extension data: } \mathfrak{a}_3 = \langle e_1, e_2, e_3 \rangle, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathfrak{s}_{5,3}$

$$[e_5, e_2] = e_1, [e_5, e_3] = e_3, [e_5, e_4] = ae_4, \text{ where } 0 < |a| \leq 1. \text{ If } |a| = 1 \text{ then } \arg(a) \leq \pi.$$

$$\text{Double extension data: } \mathfrak{a}_3 = \langle e_1, e_2, e_3 \rangle, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & a \end{pmatrix}.$$

$\mathfrak{s}_{5,4}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,3}$)

$$[e_5, e_2] = e_1, [e_5, e_3] = \alpha e_3 - e_4, [e_5, e_4] = e_3 + \alpha e_4, \text{ where } 0 \leq \alpha.$$

$$\text{Double extension data: } \mathfrak{a}_3 = \langle e_1, e_2, e_3 \rangle, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & -1 & \alpha \end{pmatrix}.$$

$\mathfrak{s}_{5,5}$

$$[e_5, e_1] = e_1, [e_5, e_2] = e_1 + e_2, [e_5, e_3] = e_2 + e_3, [e_5, e_4] = e_3 + e_4.$$

$$\text{Double extension data: } \mathfrak{a}_3 = \langle e_1, e_2, e_3 \rangle, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathfrak{s}_{5,6}$

$$[e_5, e_1] = e_1, [e_5, e_2] = e_1 + e_2, [e_5, e_3] = ae_3, [e_5, e_4] = e_3 + ae_4,$$

$$\text{where } 0 < |a| \leq 1, \text{ if } |a| = 1 \text{ then } \arg(a) \leq \pi.$$

$$\text{Double extension data: } \mathfrak{a}_3 = \langle e_1, e_2, e_3 \rangle, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{pmatrix}.$$

$\mathfrak{s}_{5,7}$

$$[e_5, e_1] = e_1, [e_5, e_2] = e_1 + e_2, [e_5, e_3] = e_2 + e_3, [e_5, e_4] = ae_4, \text{ where } a \neq 0.$$

Double extension data: $\mathfrak{a}_3 = \langle e_1, e_2, e_3 \rangle, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}.$

$\mathfrak{s}_{5,8}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,6}$)

$[e_5, e_1] = \alpha e_1 - e_2, [e_5, e_2] = e_1 + \alpha e_2, [e_5, e_3] = e_1 + \alpha e_3 - e_4, [e_5, e_4] = e_2 + e_3 + \alpha e_4,$
where $0 \leq \alpha$.

Double extension data: $\mathfrak{a}_3 = \langle e_1, e_2, e_3 \rangle, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} \alpha & 1 & 1 & 0 \\ -1 & \alpha & 0 & 1 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & -1 & \alpha \end{pmatrix}.$

$\mathfrak{s}_{5,9}$

$[e_5, e_1] = e_1, [e_5, e_2] = ae_2, [e_5, e_3] = be_3, [e_5, e_4] = ce_4$, where the value of the parameters a, b, c are $0 < |c| \leq |b| \leq |a| \leq 1$. If one or more equalities hold further restrictions on the parameters are necessary in order to avoid redundancies (for more details see [15]).

Double extension data: $\mathfrak{a}_3 = \langle e_1, e_2, e_3 \rangle, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}.$

$\mathfrak{s}_{5,10}$

$[e_5, e_1] = e_1, [e_5, e_2] = e_1 + e_2, [e_5, e_3] = ae_3, [e_5, e_4] = be_4$, where $0 < |b| \leq |a|$, if $|b| = |a|$ then $\arg(a) \leq \arg(b)$.

Double extension data: $\mathfrak{a}_3 = \langle e_1, e_2, e_3 \rangle, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}.$

$\mathfrak{s}_{5,11}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,9}$)

$[e_5, e_1] = \alpha e_1, [e_5, e_2] = \beta e_2, [e_5, e_3] = \gamma e_3 - e_4, [e_5, e_4] = e_3 + \gamma e_4$, where $\alpha > 0, \beta \neq 0, \alpha \geq \beta \geq -\alpha$.

Double extension data: $\mathfrak{a}_3 = \langle e_1, e_2, e_3 \rangle, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 1 \\ 0 & 0 & -1 & \gamma \end{pmatrix}.$

$\mathfrak{s}_{5,12}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,10}$)

$[e_5, e_1] = e_1, [e_5, e_2] = e_1 + e_2, [e_5, e_3] = \alpha e_3 - \beta e_4, [e_5, e_4] = \beta e_3 + \alpha e_4$, where $\beta > 0$

Double extension data: $\mathfrak{a}_3 = \langle e_1, e_2, e_3 \rangle, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{pmatrix}$

$\mathfrak{s}_{5,13}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{sl}_{5,9}$)

$[e_5, e_1] = \alpha e_1 - e_2, [e_5, e_2] = e_1 + \alpha e_2, [e_5, e_3] = \beta e_3 - \gamma e_4, [e_5, e_4] = \gamma e_3 + \beta e_4$, where $0 < \gamma \leq 1, 0 \leq \beta$. If $\alpha = 0$ then $0 \leq \beta$. If $\gamma = 1$ then $|\alpha| \leq |\beta|$.

Double extension data: $\mathfrak{n}_3 = \langle e_1, e_2, e_3 \rangle, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} \alpha & 1 & 0 & 0 \\ 1 & \alpha & 0 & 0 \\ 0 & 0 & \beta & \gamma \\ 0 & 0 & -\gamma & \beta \end{pmatrix}.$

$\mathfrak{s}_{5,14}$

$$[e_2, e_3] = e_1, [e_5, e_3] = e_2, [e_5, e_4] = e_4.$$

Double extension data: $\mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

$\mathfrak{s}_{5,15}$

$$[e_2, e_3] = e_1, [e_5, e_2] = e_2, [e_5, e_3] = -e_3, [e_5, e_4] = e_1.$$

Double extension data: $\mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

$\mathfrak{s}_{5,16}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,15}$)

$$[e_2, e_3] = e_1, [e_5, e_2] = -e_3, [e_5, e_3] = e_2, [e_5, e_4] = e_1.$$

Double extension data: $\mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

$\mathfrak{s}_{5,17}$

$[e_2, e_3] = e_1, [e_5, e_2] = e_2, [e_5, e_3] = -e_3, [e_5, e_4] = ae_4$, where the values of the parameter a are: over the field $\mathbb{F} = \mathbb{C}$: $0 \leq \text{Re}(a)$, if $\text{Re}(a) = 0$ then $0 < \text{Im}(a)$; over the field $\mathbb{F} = \mathbb{R}$: $0 < a$.

Double extension data: $\mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}.$

$\mathfrak{s}_{5,18}$

$$[e_2, e_3] = e_1, [e_5, e_2] = -e_2, [e_5, e_3] = e_3 + e_4, [e_5, e_4] = e_4$$

Double extension data: $\mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$

$\mathfrak{s}_{5,19}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,19}$)

$$[e_2, e_3] = e_1, [e_5, e_2] = -e_3, [e_5, e_3] = e_2, [e_5, e_4] = \alpha e_4 \text{ where } \alpha > 0.$$

Double extension data: $\mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}.$

$\mathfrak{s}_{5,20}$

$$[e_2, e_3] = e_1, [e_5, e_1] = e_1, [e_5, e_2] = e_2, [e_5, e_3] = e_4.$$

Double extension data: $\mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$

$\mathfrak{s}_{5,21}$

$$[e_2, e_3] = e_1, [e_5, e_1] = 2e_1, [e_5, e_2] = e_2 + e_3, [e_5, e_3] = e_3 + e_4, [e_5, e_4] = e_4.$$

Double extension data: $\mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$

$\mathfrak{s}_{5,22}$

$[e_2, e_3] = e_1, [e_5, e_1] = (a+1)e_1, [e_5, e_2] = e_2, [e_5, e_3] = ae_3, [e_5, e_4] = be_4$, where the value of the parameters a, b are over the field $\mathbb{F} = \mathbb{C}$: $0 < |a| \leq 1, b \neq 0$, if $|a| = 1$ then $\arg(a) < \pi$; over the field $\mathbb{F} = \mathbb{R}$: $-1 < a \leq 1, a, b \neq 0$.

Double extension data: $\mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} a+1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}.$

$\mathfrak{s}_{5,23}$

$[e_2, e_3] = e_1, [e_5, e_1] = (a+1)e_1, [e_5, e_2] = ae_2, [e_5, e_3] = e_3 + e_4, [e_5, e_4] = e_4$ where $a \neq 0, -1$.

Double extension data: $\mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} a+1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$

$\mathfrak{s}_{5,24}$

$[e_2, e_3] = e_1, [e_5, e_1] = 2e_1, [e_5, e_2] = e_2 + e_3, [e_5, e_3] = e_3, [e_5, e_4] = ae_4$ where $a \neq 0$.

Double extension data: $\mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}.$

$\mathfrak{s}_{5,25}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,22}$)

$[e_2, e_3] = e_1, [e_5, e_1] = 2\alpha e_1, [e_5, e_2] = \alpha e_2 - e_3, [e_5, e_3] = e_2 + \alpha e_3, [e_5, e_4] = \beta e_4$ where $\alpha \neq 0, \beta > 0$.

Double extension data: $\mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 2\alpha & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & -1 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}.$

$\mathfrak{s}_{5,26}$

$[e_2, e_3] = e_1, [e_5, e_1] = (a+1)e_1, [e_5, e_2] = e_2, [e_5, e_3] = ae_3, [e_5, e_4] = e_1 + (a+1)e_4$, where the values of the parameter a are over $\mathbb{F} = \mathbb{C}$: $0 < |a| \leq 1$, if $|a| = 1$ then $\arg(a) < \pi$; over the field $\mathbb{F} = \mathbb{R}$: $-1 < a \leq 1, a \neq 0$.

Double extension data: $\mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} a+1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a+1 \end{pmatrix}.$

$\mathfrak{s}_{5,27}$

$[e_2, e_3] = e_1, [e_5, e_1] = 2e_1, [e_5, e_2] = e_2 + e_3, [e_5, e_3] = e_3, [e_5, e_4] = e_1 + 2e_4.$

Double extension data: $\mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$

$\mathfrak{s}_{5,28}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,26}$)

$$[e_2, e_3] = e_1, [e_5, e_1] = 2\alpha e_1, [e_5, e_2] = \alpha e_2 + e_3, [e_5, e_3] = -e_2 + \alpha e_3, [e_5, e_4] = e_1 + 2\alpha e_4, \text{ where } \alpha > 0.$$

$$\text{Double extension data: } \mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 2\alpha & 0 & 0 & 1 \\ 0 & \alpha & -1 & 0 \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & 0 & 2\alpha \end{pmatrix}.$$

$\mathfrak{s}_{5,29}$

$$[e_2, e_3] = e_1, [e_5, e_1] = e_1, [e_5, e_3] = e_3 + e_4, [e_5, e_4] = e_4.$$

$$\text{Double extension data: } \mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

$\mathfrak{s}_{5,30}$

$$[e_2, e_3] = e_1, [e_5, e_1] = e_1, [e_5, e_2] = e_2, [e_5, e_4] = ae_4, \text{ where } a \neq 0.$$

$$\text{Double extension data: } \mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}.$$

$\mathfrak{s}_{5,31}$

$$[e_2, e_3] = e_1, [e_5, e_1] = e_1, [e_5, e_2] = e_2, [e_5, e_4] = e_1 + e_4.$$

$$\text{Double extension data: } \mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathfrak{s}_{5,32}$

$$[e_2, e_3] = e_1, [e_5, e_1] = e_1, [e_5, e_3] = e_3 + e_4, [e_5, e_4] = e_1 + e_4.$$

$$\text{Double extension data: } \mathfrak{n}_{3,1} = \langle e_1, e_2, e_3 \rangle, [e_2, e_3] = e_1, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

$\mathfrak{s}_{5,33}$

$$[e_2, e_4] = e_1, [e_3, e_4] = e_2, [e_5, e_2] = -e_2, [e_5, e_3] = -2e_3, [e_5, e_4] = e_4.$$

$$\text{Double extension data: } \mathfrak{n}_{3,1} = \langle e_2, e_3, e_4 \rangle, [e_3, e_4] = e_2, \theta_{24} = 1, \text{ad}(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathfrak{s}_{5,34}$

$$[e_2, e_4] = e_1, [e_3, e_4] = e_2, [e_5, e_1] = 3e_1, [e_5, e_2] = 2e_2, [e_5, e_3] = e_3, [e_5, e_4] = e_3 + e_4.$$

$$\text{Double extension data: } \mathfrak{n}_{3,1} = \langle e_2, e_3, e_4 \rangle, [e_3, e_4] = e_2, \theta_{24} = 1, \text{ad}(e_5) = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathfrak{s}_{5,35}$

$$[e_2, e_4] = e_1, [e_3, e_4] = e_2, [e_5, e_1] = (a+2)e_1, [e_5, e_2] = (a+1)e_2, [e_5, e_3] = ae_3, [e_5, e_4] = e_4, \text{ where } a \neq 0, -2.$$

$$\text{Double extension data: } \mathfrak{n}_{3,1} = \langle e_2, e_3, e_4 \rangle, [e_3, e_4] = e_2, \theta_{24} = 1, \text{ad}(e_5) = \begin{pmatrix} a+2 & 0 & 0 & 0 \\ 0 & a+1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathfrak{s}_{5,36}$

$$[e_2, e_4] = e_1, [e_3, e_4] = e_2, [e_5, e_1] = 2e_1, [e_5, e_2] = e_2, [e_5, e_4] = e_4.$$

$$\text{Double extension data: } \mathfrak{n}_{3,1} = \langle e_2, e_3, e_4 \rangle, [e_3, e_4] = e_2, \theta_{24} = 1, \text{ad}(e_5) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

 $\mathfrak{s}_{5,37}$

$$[e_2, e_4] = e_1, [e_3, e_4] = e_2, [e_5, e_1] = e_1, [e_5, e_2] = e_2, [e_5, e_3] = e_3.$$

$$\text{Double extension data: } \mathfrak{n}_{3,1} = \langle e_2, e_3, e_4 \rangle, [e_3, e_4] = e_2, \theta_{24} = 1, \text{ad}(e_5) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

 $\mathfrak{s}_{5,38}$

$[e_2, e_4] = e_1, [e_3, e_4] = e_2, [e_5, e_1] = e_1, [e_5, e_2] = e_2, [e_5, e_3] = \epsilon e_1 + e_3$ where the values of the parameter ϵ are over the field $\mathbb{F} = \mathbb{C}$: $\epsilon = 1$; over the field $\mathbb{F} = \mathbb{R}$: $\epsilon = \pm 1$.

$$\text{Double extension data: } \mathfrak{n}_{3,1} = \langle e_2, e_3, e_4 \rangle, [e_3, e_4] = e_2, \theta_{24} = 1, \text{ad}(e_5) = \begin{pmatrix} 1 & 0 & \epsilon & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

 $\mathfrak{s}_{5,39}$

$$[e_4, e_2] = e_2, [e_5, e_3] = e_3, [e_5, e_4] = e_1.$$

$$\text{Double extension data: } \mathfrak{n}_{2,1} \oplus \langle e_1 \rangle = \langle e_2, e_3, e_4 \rangle, [e_4, e_2] = e_2, \theta = 0,$$

$$\text{ad}(e_5) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

 $\mathfrak{s}_{5,40}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,39}$)

$$[e_4, e_2] = e_2, [e_4, e_3] = e_3, [e_5, e_2] = -e_3, [e_5, e_3] = e_2, [e_5, e_4] = e_1.$$

$$\text{Double extension data: } \mathfrak{q}_1 = \langle e_2, e_3, e_4 \rangle, [e_4, e_2] = e_2, [e_4, e_3] = e_3, \theta = 0,$$

$$\text{ad}(e_4) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

 $\mathfrak{s}_{5,41}$

$[e_4, e_1] = e_1, [e_4, e_3] = ae_3, [e_5, e_2] = e_2, [e_5, e_3] = be_3$, where the values of the parameters a, b are $0 < |b| \leq |a| \leq 1$. If one or more equalities hold further restrictions are necessary on the parameters are necessary to avoid redundancies (for more details see [15]).

$$\text{Double extension data: } \mathfrak{q}_a = \langle e_1, e_3, e_4 \rangle, [e_4, e_1] = e_1, [e_4, e_3] = ae_3, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

 $\mathfrak{s}_{5,42}$

$$[e_4, e_1] = ae_1, [e_4, e_2] = e_2, [e_4, e_3] = e_3, [e_5, e_1] = e_2, [e_5, e_3] = e_2$$

Double extension data: - -.

$\mathfrak{s}_{5,43}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,41}$)

$$[e_4, e_1] = \alpha e_1, [e_4, e_2] = e_2, [e_4, e_3] = e_3, [e_5, e_1] = \beta e_1, [e_5, e_2] = -e_3, [e_5, e_3] = e_2, \text{ where } \alpha^2 + \beta^2 \neq 0.$$

Double extension data: - -.

$\mathfrak{s}_{5,44}$

$$[e_4, e_1] = e_1, [e_4, e_2] = e_2, [e_5, e_2] = e_2, [e_5, e_3] = -e_3$$

$$\text{Double extension data: } q_1 = \langle e_1, e_2, e_4 \rangle, [e_4, e_1] = e_1, [e_4, e_2] = e_2, \theta = 0, \text{ad}(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

ACKNOWLEDGEMENT

The authors would like to thank the referee for taking the time and effort necessary to review the manuscript. We sincerely appreciate all valuable comments and suggestions, which helped us to improve the quality of the manuscript.

The authors would like to acknowledge the support received by PRODEP grant UASLP-CA-228 and CONACyT grant 222870.

REFERENCES

- [1] M. A. Alvarez, M. C. Rodríguez-Vallarte, G. Salgado, *Contact Nilpotent Lie algebras*, Proc. Amer. Math. Soc. **145**(2017), no. 4, 1467-1474. <https://doi.org/10.1090/proc/1334>
- [2] M. A. Alvarez, M. C. Rodríguez-Vallarte, G. Salgado, *Deformation theory of contact Lie algebras as double extensions*. Proc. Amer. Math. Soc. **149**(2021), no. 5, 1827-1836. <https://doi.org/10.2307/1970165>
- [3] I. Bajo, S. Benayadi, A. Medina, *Symplectic structures on quadratic Lie algebras*, J. Algebra **316**(2007), 174-188. <https://doi.org/10.1016/j.jalgebra.2007.06.001>
- [4] S. Benayadi, *A new characterization of semisimple Lie algebras*, Proc. Amer. Math. Soc. **125**(1997), no. 3, 685-688. <https://doi.org/10.1090/S0002-9939-97-03612-5>
- [5] L. Boza, E. M. Fedriani, J. Núñez, A. F. Tenorio, *A historical review of the classifications of Lie algebras*, Rev. Un. Mat. Argentina **54**(2013), no. 2, 75-99.

- [6] C. Chevalley, S. Eilenberg, *Cohomology Theory of Lie Groups and Lie Algebras*. Trans. Amer. Math. Soc. **63**(1948), 85-124. <https://doi.org/10.1090/S0002-9947-1948-0024908-8>
- [7] J. E. Humphreys, *Introduction to Lie algebras and Representation Theory*, Graduate Texts in Mathematics, Vol. 9. Springer-Verlag, New York-Berlin, 1972. <https://doi.org/10.1007/978-1-4612-6398-2>
- [8] N. Jacobson, *Lie algebras*, Republication of the 1962 original. Dover Publications, Inc., New York, 1979.
- [9] V. Kac, *Infinite-dimensional Lie algebras*, Cambridge University Press, Cambridge, 1990. <https://doi.org/10.1017/CBO9780511626234>
- [10] A. Medina, P. Revoy, *Algèbres de Lie et produit scalaire invariant* (In French), Ann. Scient. École Norm. Sup. 4 **18**(1985), no. 3, 553-561.
- [11] A. Medina, P. Revoy, *Groupes de Lie à structure symplectique invariante*, Math. Sci. Res. Inst. Publ. **20**(1991), 247-266. https://doi.org/10.1007/978-1-4613-9719-9_17
- [12] A. L. Onishchik, E. B. Vinberg, *Lie Groups and Algebraic Groups*, Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990. <https://doi.org/10.1007/978-3-642-74334-4>
- [13] M. C. Rodríguez-Vallarte, G. Salgado, *5-dimensional indecomposable contact Lie algebras as double extensions*, J. Geom. Phys. **100**(2016), 20-32. <https://doi.org/10.1016/j.geomphys.2015.10.014>
- [14] G. Salgado, C. Villalobos-Guillén, *Reductive and semisimple Lie algebras; new characterizations* (In Spanish), Aportaciones Matemáticas, Mem. Soc. Mat. Mex. **13**(2017), 113-122.
- [15] L. Šnobl, P. Winternitz, *Classification and Identification of Lie algebras*, Vol. 33. American Mathematical Soc. 2017. <https://doi.org/10.1090/crmm/033>

FAC. DE CIENCIAS, UASLP, AV. PARQUE CHAPULTEPEC 1570, PRIVADAS DEL PEDREGAL, CP 78210, SAN LUIS POTOSÍ, S.L.P., MÉXICO.

E-mail address: carmen.vallarte@uaslp.mx, mcvallarte@fc.uaslp.mx,

E-mail address: gsalgado@fciencias.uaslp.mx, gil.salgado@gmail.com

La Sociedad Matemática Mexicana

Junta Directiva de la Sociedad Matemática Mexicana 2024-2026:

Dra. Martha Gabriela Araujo Pardo	Presidenta
Dr. Alvaro Eduardo Cordero Franco	Vicepresidente
Dr. Víctor Castellanos Vargas	Secretario General
Dra. María del Carmen Rodríguez Vallarte	Tesorera
Dr. Ricardo Alberto Sáenz Casas	Secretario de Actas
Dra. Ivette Sánchez Bravo	Secretaria de Vinculación
Dra. Natalia Bárbara Mantilla Beniers	Vocal
Dr. Luis Miguel García Velázquez	Vocal
Dra. María Guadalupe Russell Noriega	Miembro Comité Consultivo
Dr. Adrián González Casanova	Miembro Comité Consultivo
Dr. José Fernando Camacho Vallejo	Miembro Comité Consultivo

La Sociedad Matemática Mexicana fue fundada en el año de 1943, por lo que cumplió 80 años de existencia este 2024. Entre sus tareas fundamentales destacan: estimular y mantener el interés por la investigación matemática en México; publicar revistas científicas; contribuir al mejoramiento de los programas de estudio y la enseñanza de las matemáticas en el país; estimular la elaboración y publicación de libros de texto en matemáticas; organizar conferencias, reuniones, congresos y concursos de matemáticas; cooperar en la resolución de los problemas matemáticos que se presentan en el sector productivo del país, así como los que surgen en las investigaciones de otras ciencias; promover el convencimiento entre la sociedad mexicana de que las matemáticas son parte integral de la cultura básica y de las habilidades de uso cotidiano de la población.

www.smm.org.mx

Instituciones patrocinadoras de la Sociedad Matemática Mexicana:

- Centro de Ciencias Matemáticas, UNAM, Campus Morelia
- Centro de Investigación en Matemáticas A.C.
- Centro de Investigación y de Estudios Avanzados del IPN
- Instituto de Matemáticas, UNAM
- Universidad de Sonora

El Instituto de Matemáticas de la UNAM

La misión básica del Instituto de Matemáticas de la UNAM es realizar investigación en matemáticas del más alto nivel, así como en una variedad de áreas lo más amplia posible, con el objetivo de que las matemáticas en México lleguen a desempeñar el papel estratégico que de ellas exige la sociedad contemporánea. Como parte consustancial de lo anterior, y en sintonía con el compromiso de la UNAM con el desarrollo del país, esta misión implica hacer un importante énfasis en la formación de recursos humanos y en la enseñanza a todos los niveles, incluyendo la divulgación.

El Instituto ha logrado desarrollar esta amplia gama de actividades de manera integral. A lo largo de sus más de 80 años de historia ha sabido sembrar la semilla de las matemáticas en otras partes de la República. Su apoyo a la formación de otras instituciones de investigación y enseñanza ha sido siempre decidido, en concordancia con los principios de nuestra máxima casa de estudios. Tan sólo en lo que se refiere al ámbito de la UNAM, destacamos la creación de tres Unidades Foráneas: Cuernavaca, Juriquilla y Oaxaca.

www.matem.unam.mx

Directorio del Instituto de Matemáticas

- | | |
|-----------------------------|------------------------------------|
| ▪ Dirección | Dra. Isabel Alicia Hubard Escalera |
| ▪ Secretaría Académica | Dr. Ricardo Gómez Aiza |
| ▪ Secretaría Administrativa | Lic. Jorge González Enríquez |

Sección de Publicaciones: Se encarga de la edición, difusión y distribución de las publicaciones periódicas, colecciones, series y textos que se elaboran en el Instituto.

Dr. Juan José Montellano Ballesteros (coordinador de la Sección de Publicaciones)

Fís. Helena Lluís Arroyo (responsable administrativa, de convenios, corrección de estilo, edición y lectura de pruebas especializadas)

Dr. Pablo Rosell González (responsable de diseño gráfico y diseño editorial, consultoría y revisión especializadas)

Leonardo Espinosa Pérez (responsable técnico de edición, consultoría y revisión especializadas)

Celia Osorio Martínez (colaboración en inventarios, control de ventas y apoyo logístico)

Yohali Hernández López (colaboración en inventarios, ventas y apoyo logístico)



**SOCIEDAD
MATEMÁTICA
MEXICANA**



Instituto de
Matemáticas UNAM